A robust grid equidistribution method for a one-dimensional
singularly perturbed semilinear reaction-diffusion problem∗

NARESH M. CHADHA AND NATALIA KOPTEVA†

Mathematics and Statistics Department, University of Limerick,
Limerick, Ireland.

The numerical solution of a singularly perturbed semilinear reaction-diffusion two-point boundary value
problem is addressed. The method considered is adaptive movement of a fixed number \((N + 1)\) of mesh
points by equidistribution of a monitor function that uses discrete second-order derivatives. We extend the
analysis by Kopteva & Stynes (2001) to a new equation and a more intricate monitor function. It is proved
that there exists a solution to the fully discrete equidistribution problem, i.e. a mesh exists that equidis-
tributes the discrete monitor function computed from the discrete solution on this mesh. Furthermore, in
the case when the boundary value problem is linear, it is shown that after \(O(|\ln \varepsilon|/\ln N)\) iterations of the
algorithm, the piecewise linear interpolant of the computed solution achieves second-order accuracy in
the maximum norm, uniformly in the diffusion coefficient \(\varepsilon^2\). Numerical experiments are presented that
support our theoretical results.

Keywords: grid equidistribution, reaction-diffusion, singular perturbation, adaptive mesh, finite differ-
ences, maximum norm.

1. Introduction

Solutions of singularly perturbed differential equations frequently exhibit sharp boundary and interior
layers, which are narrow regions where solutions change rapidly. To obtain reliable numerical approxi-
mations of layer solutions in an efficient way, one has to use locally refined meshes that are fine in layer
regions and standard outside. If the location(s) and width(s) of the layers are known a priori, one can
invoke this knowledge to construct suitable layer-adapted meshes. Otherwise, which is often the case
in applications, adaptive algorithms are needed that start from an initial unsophisticated mesh and then,
using intermediate computed solutions, automatically adapt the mesh and ultimately detect accurate lo-
cations and widths of the layers. Although adaptive algorithms have been successfully applied to many
problems, their convergence properties are not entirely understood.

The aim of this paper is to present a convergence analysis of an adaptive algorithm applied to one
simple singularly perturbed problem. We follow Kopteva & Stynes (2001), where a similar algorithm,
which equidistributed the arc-length of the computed solution, was applied to a singularly perturbed
convection-diffusion equation and the number of iterations was estimated needed to achieve first-order
accuracy. Now we extend the analysis in Kopteva & Stynes (2001) to a new equation and a more intri-
cate monitor function that involves discrete second-order derivatives and whose equidistribution yields
second-order-accurate computed solutions. Compared to the arc-length monitor function considered in
Kopteva & Stynes (2001), our monitor function yields higher-order accuracy, but its equidistribution
does not allow a simple geometric interpretation and hence its analysis is more complicated.

∗This publication has emanated from research conducted with the financial support of Science Foundation Ireland under the
Basic Research Grant Programme 2004; Grant 04/BR/M0055.
†Email: Naresh.Chadha@ul.ie; Natalia.Kopteva@ul.ie
We shall address the singularly perturbed semilinear reaction-diffusion problem
\[ Tu = -\varepsilon^2 u'''(x) + b(x, u) = 0 \quad \text{for } x \in (0, 1), \quad u(0) = u(1) = 0, \]  
(1.1)

where \( \varepsilon \in (0, 1) \) is a small positive parameter, the function \( b \) is sufficiently smooth and
\[ 0 < \gamma_0^2 \leq b_{\varepsilon}(x, u) \leq \gamma^2 \quad \text{for all } (x, u) \in [0, 1] \times \mathbb{R}. \]  
(1.2)

Under condition (1.2), problem (1.1) has a unique solution, which exhibits sharp boundary layers of width \( O(\varepsilon \ln \varepsilon) \) at \( x = 0 \) and \( x = 1 \).

For our numerical method we consider arbitrary meshes \( \{x_i\}_{i=0}^N \) with \( 0 = x_0 < x_1 < \ldots < x_N = 1 \), where \( N \) is a positive integer, and the local mesh size \( h_i := x_i - x_{i-1} \) for \( i = 1, \ldots, N \). The computed solution \( \{u_N^i\}_{i=0}^N \) associated with the mesh \( \{x_i\}_{i=0}^N \), is required to satisfy the standard finite difference discretization of (1.1):
\[ T^N u_N^i := -\varepsilon^2 \delta^2 u_N^i + b(x_i, u_N^i) = 0 \quad \text{for } i = 1, \ldots, N - 1, \quad u_N^0 = u_N^N = 0, \]  
(1.3a)

where
\[ \delta^2 u_N^i := \frac{2}{h_i + h_{i+1}} \left( u_{i+1}^{N} - u_i^{N} - \frac{u_i^{N} - u_{i-1}^{N}}{h_i} \right), \quad i = 1, \ldots, N - 1. \]  
(1.3b)

Note that there exists a unique solution \( \{u_N^i\} \) to the discrete problem (1.3); furthermore, a linearization of this problem satisfies the discrete maximum/comparison principle.

We shall invoke the a posteriori error analysis by Kopteva (2007). In particular, for the solution \( u \) of (1.1) and the standard piecewise linear interpolant \( u_N^i(x) \) of the discrete solution \( \{u_N^i\} \) of (1.3) on an arbitrary mesh \( \{x_i\} \), Kopteva (2007) gives the following a posteriori error estimate:
\[ \max_{0 < x < 1} |u_N^i(x) - u(x)| \leq C \max_{1 \leq i \leq N} \{M_N h_i\}^2, \quad M_N := \min \{|\delta^2 u_{i-1}^{N}|, |\delta^2 u_i^{N}|\}^{1/2} + (\varepsilon |\delta^1 u_i^{N}|)^{1/2} + 1, \]  
(1.4a)

which also uses the standard third-order discrete derivatives \( \delta^1 u_i^{N} := (\delta^2 u_i^{N} - \delta^2 u_{i-1}^{N})/h_i \) (combine Theorem 3.3, Remark 3.4 and Lemma 3.5 in Kopteva (2007)). Here \( M_N h_i \) and \( M_N \) involve \( \delta^2 u_i^{N} \) for \( i = 0, N \), respectively, which are not given in (1.3b), but instead defined by
\[ \delta^2 u_0^{N} := \varepsilon^{-2} b(0, u_0^{N}), \quad \delta^2 u_N^{N} := \varepsilon^{-2} b(1, u_N^{N}), \]  
(1.4b)
in agreement with the discrete equation \(-\varepsilon^2 \delta^2 u_N^i + b(x_i, u_N^i) = 0 \) from (1.3a) formally extended to \( i = 0, N \). It is crucial that the error constant \( C \) in (1.4) is independent of \( \varepsilon \) and the mesh. We also refer the reader to a two-dimensional version of the a posteriori error estimate (1.4) recently presented in Kopteva (2007b).

Furthermore, it is shown in (Kopteva, 2007, Lemma 3.5) that on an arbitrary mesh we have
\[ \max_{1 \leq i \leq N} \{M_N h_i\} \leq C \max_{1 \leq i \leq N} \{M_N h_i\}, \quad M_N := \max \{|\delta^2 u_{i-1}^{N}|, |\delta^2 u_i^{N}|\}^{1/2} + 1. \]  
(1.5)

This implies that the discrete analogue of \( \varepsilon |u'''| \) in the a posteriori error estimate (1.4a) is insignificant.

Given the a posteriori error estimate (1.4) and relation (1.5), we face the problem of finding the mesh \( \{x_i\} \) and the computed solution \( \{u_N^i\} \) on this mesh such that \( \max_{1 \leq i \leq N} M_N h_i \leq CN^{-1} \) or \( \max_{1 \leq i \leq N} M_N h_i \leq CN^{-1} \); then, by (1.4), (1.5), this computed solution \( \{u_N^i\} \) is guaranteed to be second-order accurate \( \varepsilon \)-uniformly.
Grid equidistribution for a singularly perturbed reaction-diffusion problem

We shall address this problem in the framework of monitor-function equidistribution. A mesh \( \{x_i\} \) is said to equidistribute a monitor function \( M(x) > 0 \) if

\[
\int_{x_{i-1}}^{x_i} M(x) \, dx = \frac{1}{N} \int_0^1 M(x) \, dx \quad \text{for} \quad i = 1, \ldots, N. \tag{1.6}
\]

The a posteriori error estimate (1.4) and relation (1.5) imply a piecewise constant monitor function \( M(x) = \bar{M}_N \) for \( x \in (x_{i-1}, x_i) \), where \( M_N^i = \bar{M}_N^i \) or \( M_N^i = \bar{M}_N^i \), which yields the following formulation.

**Equidistribution Problem.** Find \( \{x_i, u_N^i\} \), with the \( \{u_N^i\} \) computed from the \( \{x_i\} \) by means of (1.3), such that

\[
M_N^i h_i = \frac{1}{N} \sum_{j=1}^{N} M_N^j h_j \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{1.7}
\]

Note that here both \( \{x_i\} \) and \( \{u_N^i\} \) are a priori unknown. Consequently, even if (1.1) is linear, the equidistribution problem, which requires the simultaneous solution of (1.3) and (1.7), is nonlinear. Following the analysis by Kopteva & Stynes (2001) of arc-length equidistribution for a convection-diffusion equation, we pose the following fundamental questions:

1. Does our equidistribution problem have a solution?
2. Is there any algorithm to solve our equidistribution problem that can be proved to yield an accurate computed solution?

The aim of the present paper is to provide answers to these important questions:

1. We establish existence of a solution to the equidistribution problem (1.7). Furthermore, our existence theorem applies to a more general equidistribution problem that uses an abstract discrete problem and an abstract monitor function (not necessarily discrete problem (1.3) and the monitor function \( \bar{M}_N^i \) from (1.5) or \( \bar{M}_N^i \) from (1.8) below), which are required to satisfy a certain basic condition; see Theorem 3.1 for details. Furthermore, the abstract discrete problem might be a discretization of a continuous problem other than our problem (1.1).

2. We consider a simple algorithm, originally due to de Boor (1974), that equidistributes the discrete monitor function

\[
M_N^i := \min\{\|\delta^2 u_N^{i-1}\|, \|\delta^2 u_N^i\|\}^{1/2} + 1, \quad \text{where} \quad \delta^2 u_N^0 := \delta^2 u_N^1, \quad \delta^2 u_N^N := \delta^2 u_N^{N-1}. \tag{1.8}
\]

We choose this modification of \( \bar{M}_N^i \) for our algorithm, since using \( M_N^i \) from (1.4) or \( \bar{M}_N^i \) from (1.5) instead of this \( M_N^i \) might result in the so-called mesh starvation when the mesh nodes are temporarily drawn away from certain regions; see Figure 2 in §6 and Remarks 2.3 and 4.5.

Following Kopteva & Stynes (2001), de Boor’s original algorithm is modified by using a less stringent stopping criterion that, while preserving the accuracy of the final computed solution, requires far fewer iterations.

We answer question 2 in the case when (1.1) is linear and under further mild assumptions. Then it is stated in Theorem 5.1 that our algorithm is guaranteed to yield a second-order accurate computed solution and this is achieved after \( O(\ln \epsilon / \ln N) \) iterations.

Note that we expect any other efficient algorithm based on a similar monitor function to enjoy similar properties and require a similar number of iterations.
Monitor functions have been used by many authors to drive adaptive algorithms in solving differential equations; see, e.g., Huang & Sloan (1994), Cao et al. (1999), Linß (2001), Tang & Tang (2003), Kopteva et al. (2005), Huang (2005a), Huang (2005b), Mackenzie & Mekwi (2007). In particular, the monitor function $M_N^{a}$ used in (1.7) is a discrete analogue of the continuous monitor function $M(x) := |u''|^{1/2} + 1$, whose equidistribution (1.6) for a linear analogue of (1.1) was investigated by Beckett & Mackenzie (2001). Note also that the same continuous monitor function $M(x)$ is a one-dimensional version of the monitor function that was suggested by Chen et al. (2007) based on the piecewise-linear-interpolation error analysis in the $L_p$ norm in the case of $p = \infty$.

Outline. The mesh movement algorithm is described in §2. Next, in §3 we prove our existence result (Theorem 3.1). Then in §4 we establish an important property of any intermediate mesh generated by our algorithm that its maximum mesh size never exceeds $CN^{-1}$. The entire §5 is devoted to a detailed analysis of the algorithm’s behaviour in the case of linear $b(x,u)$; this culminates in an estimate of Theorem 5.1 for the number of iterations needed to achieve second-order accuracy. Numerical results supporting our theory are presented in §6. Note that to facilitate reading this paper, we deliberately follow the notation and presentation in Kopteva & Stynes (2001), which gave a similar analysis but for a simpler monitor function. We also refer the reader to Kopteva (2007a) for a constant-coefficient case of the analysis in Kopteva & Stynes (2001).

Notation: Throughout the paper $C, C', \bar{C}$ denote generic positive constants that are independent of $\varepsilon$, of the mesh, and of the number of iterations taken by the algorithm of §2, and can take different values in different places. A subscripted $C$ (e.g., $C_1$) is a constant that is independent of $\varepsilon$, of the mesh, and of the number of iterations taken by the algorithm, but whose value is fixed. When choosing $N$ sufficiently large independently of $\varepsilon$ and the mesh, we shall mean that $N \geq C$ for some sufficiently large positive constant $C$. As the number of mesh points $(N+1)$ is fixed throughout the algorithm, we generally do not indicate dependence on $N$ in our notation; for example we write $u^{(k)}$ for the solution computed by the $k$th iteration of the algorithm and $T^{(k)}$ for the difference operator on the $k$th mesh.

2. The algorithm

Given an arbitrary mesh, our algorithm aims to construct a mesh that solves the Equidistribution Problem (1.7), (1.8). The number $N$ of mesh intervals is fixed throughout.

ALGORITHM:

1. Initialize mesh: The initial mesh $\{0, 1/N, 2/N, \ldots, 1\}$ is uniform.

2. For $k = 0, 1, \ldots$, given the mesh $\{x^{(k)}_i\}$, compute the discrete solution $\{u^{(k)}_i\}$ satisfying

$$T^{(k)}u^{(k)} = 0 \quad \text{on} \quad \{x^{(k)}_i\}, \quad \text{with} \quad u^{(k)}_0 = u^{(k)}_N = 0. \quad (2.1)$$

Set $h^{(k)}_i = x^{(k)}_i - x^{(k)}_{i-1}$ for each $i$. Let the piecewise-constant monitor function $M^{(k)}(x)$ be defined by

$$M^{(k)}(x) := M^{(k)} = \min \left \{ \delta^2 u^{(k)}_{i-1}, |\delta^2 u^{(k)}_i| \right \}^{1/2} + 1 \quad \text{for} \quad x \in (x^{(k)}_{i-1}, x^{(k)}_i), \quad (2.2a)$$

where

$$\delta^2 u^{(k)}_0 := \delta^2 u^{(k)}_1, \quad \delta^2 u^{(k)}_N := \delta^2 u^{(k)}_{N-1}. \quad (2.2b)$$

Then the total integral of the monitor function $M^{(k)}$ is

$$I^{(k)} := \int_0^1 M^{(k)}(x) \, dx = \sum_{i=1}^{N} M^{(k)} h^{(k)}_i.$$

3. Test mesh: Let $C_0$ be a user-chosen constant with $C_0 > 1$ (see Remark 2.1). If

$$\max_{1 \leq i \leq N} \left\{ M^k_i h^k_i \right\} \leq C_0 \frac{I^k}{N},$$

(2.3)

then go to Step 5. Otherwise, continue to Step 4.

4. Generate a new mesh by equidistributing monitor function $M^k$ of current computed solution:

Choose the new mesh nodes $0 = x_0^{(k+1)} < x_1^{(k+1)} < \ldots < x_N^{(k+1)} = 1$ such that

$$\int_{x_i^{(k+1)}}^{x_{i+1}^{(k+1)}} M^k(x) \, dx = I^k / N, \quad i = 0, \ldots, N.$$  

(2.4)

(Since $\int_0^1 M^k(t) \, dt$ is increasing in $x$, the above relation clearly determines the $x_i^{(k+1)}$ uniquely.) Return to Step 2.

5. Set $\{x_0^*, x_1^*, \ldots, x_N^*\} = \{x_i^{(k)}\}$ and $\{u_i^*\} = \{u_i^{(k)}\}$ then stop.

REMARK 2.1 In (2.3) we can choose any constant $C_0$ that satisfies $C_0 > 1$. The larger $C_0$ is, the fewer iterations needed by the algorithm. If we set $C_0 = 1$, then the algorithm is attempting to compute a fixed point of Theorem 3.1, so when $C_0 \approx 1$, we expect that the computed solution lies near such a fixed point. Note that the numerical experiments presented in §6, and also in Kopteva et al. (2005), imply that $C_0 = 2$ produces suitable fary-adapted meshes and requires quite few iterations.

REMARK 2.2 The choice of $C_0 > 1$ in the stopping criterion (2.3) implies that instead of the equidistribution problem (1.7), (1.8), the algorithm attempts to solve the following Quasi-Equidistribution Problem: Find $\{(x_i, u_i^N)\}$, with the $\{u_i^N\}$ computed from the $\{x_i\}$ by means of (1.3), such that

$$M_i^N h_i \leq C_0 \frac{1}{N} \sum_{j=1}^N M_j^N h_j \quad \text{for } i = 1, 2, \ldots, N.$$  

(2.5)

REMARK 2.3 Our a posteriori error estimate (1.4) and relation (1.5) involve $\tilde{M}_i^N$ and $\tilde{M}_i^N$, so one might assume that it would be a better choice for the algorithm to use $M_i^k := \tilde{M}_i^N$ or $\tilde{M}_i^k := \tilde{M}_i^N$ instead of $M_i^k$. However, using $\tilde{M}_i^k$ and $\tilde{M}_i^k$ results in so-called mesh starvation when the mesh nodes are drawn away from certain regions; see Figure 2 in §6 and also Remark 4.5; while the algorithm in its current form enjoys the important property that $\max_i h_i^{(k)} \leq CN^{-1}$ independently of $\varepsilon$, $N$ and $k$; see Corollary 4.3.

REMARK 2.4 Instead of the monitor function $M_i^N$, it might be preferable to use its scaled version

$$M_{\beta,i}^N := \beta^{-1} \min \left\{ |\delta^2 u_{i-1}^N|, |\delta^2 u_i^N| \right\}^{1/2} + 1$$

(2.6)

for some positive constant $\beta$. This is equivalent to stretching one of the coordinates. The algorithm and its analysis can be easily modified to accommodate this more general function. Similarly, one might define $M_{\gamma,i}^N := \beta^{-1} \min \{ |\delta^2 u_{i-1}^N|, |\delta^2 u_i^N| \}^{1/2} + (\varepsilon |\delta^3 u_i^N|)^{1/2} / 2 + 1$. Furthermore, the magnitude of $u_i^N$ should not affect the suitability of the monitor function, so one natural approach is to normalize by $\beta := \max_i |u_i^N|^{1/2}$; see (Kopteva et al., 2005, relation (4.7)).
3. The existence theorem

In this section we establish existence of a solution to our equidistribution problem (1.7), (1.8). Furthermore, our existence theorem applies to a general equidistribution problem that uses an abstract discrete problem (not necessarily (1.3)) and an abstract monitor function (not necessarily $M^N$), which are required to satisfy a certain basic condition; see Theorem 3.1 below.

3.1 Abstract equidistribution problem

On an arbitrary mesh $\{x_i\}_{i=0}^N$ consider an abstract numerical method in the form

$$\hat{T}^N u_i^N = 0 \quad \text{for } i = 0, \ldots, N,$$

(3.1)

possibly involving a small parameter $\varepsilon \in (0, 1]$. Note that (3.1) might be a discretization of a continuous problem other than (1.1).

Abstract Equidistribution Problem. Find $\{(x_i, \hat{u}_i^N)\}$, with the $\{\hat{u}_i^N\}$ computed from the $\{x_i\}$ by means of (3.1), such that

$$\int_{x_{i-1}}^{x_i} \hat{M}^N(x) \, dx = \frac{1}{N} \int_{0}^{1} \hat{M}^N(x) \, dx \quad \text{for } i = 1, 2, \ldots, N,$$

(3.2)

where $\hat{M}^N(x) = \hat{M}^N(\{(x_i, \hat{u}_i^N)\}; x)$ is an abstract monitor function computed from $\{(x_i, \hat{u}_i^N)\}$ and scaled so that $\hat{M}^N(x) \geq 1$.

THEOREM 3.1 (EXISTENCE OF A SOLUTION TO THE ABSTRACT EQUIDISTRIBUTION PROBLEM) Let the abstract discrete problem (3.1) have a unique solution on each mesh $\{x_i\}$ and the associated mapping $F : \mathbb{R}^{2(N+1)} \to \mathbb{R}^{N+1}$, with $F(\hat{u}_0, \ldots, \hat{u}_N, x_0, \ldots, x_N) = \hat{T}^N \hat{u}_i^N, i = 0, \ldots, N$, be continuously differentiable and have a nonsingular Jacobian matrix $\frac{\partial F(\hat{u}_0, \ldots, \hat{u}_N)}{\partial \hat{u}_0, \ldots, \hat{u}_N}$ for all $\{\hat{u}_i^N\}$ and all $\{x_i\}$ such that $x_0 < x_1 < \ldots < x_N$.

Furthermore, suppose that there exists a quantity $Q(\varepsilon, N) \in (0, 1)$ such that the monitor function $\hat{M}^N(x)$ computed on an arbitrary mesh $\{x_i\}$ satisfies

$$1 \leq \hat{M}^N(x) \leq \frac{1}{Q(\varepsilon, N)} \quad \text{for } x \in [0, 1];$$

(3.3)

while $\int_{0}^{1} \hat{M}^N(t) \, dt$ is continuous in $\{\hat{u}_i^N\}$ and $\{x_i\}$. Then for each $\varepsilon$ and each $N$, the abstract equidistribution problem (3.2) has a solution.

Proof. We imitate the proof of (Kopteva & Stynes, 2001, Theorem 3.1), which addresses a discrete arc-length monitor function. One can regard Steps 2 and 4 of the Algorithm in §2, in which we replace $T^{(k)}$, $u^{(k)}$ and $M^{(k)}$ with $\hat{T}^N$, $\hat{u}_i^N$ and $\hat{M}^N$, respectively, as a mapping $\Phi : (h_1, h_2, \ldots, h_N) \mapsto (\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_N)$, where the $h_i$ and $\hat{h}_i$ are the mesh-widths before and after regridding. Note that $\Phi$ is continuous, since $\int_{0}^{1} \hat{M}^N(t) \, dt$ is continuous in $\{\hat{u}_i\}$ and $\{x_i\}$, while $\hat{u}_i = \hat{u}_i(x_0, \ldots, x_N), i = 0, \ldots, N$ are continuous, by the implicit function theorem applied to the system $F_i(\hat{u}_0, \ldots, \hat{u}_N, x_i, \ldots, x_N) = 0$ for $i = 0, \ldots, N$.

We claim that $\Phi : S_Q \to S_Q$, where

$$S_Q = \left\{(h_1, h_2, \ldots, h_N) \in \mathbb{R}^N : h_i \geq Q/N \text{ for } i = 1, \ldots, N, \quad \sum_{i=1}^{N} h_i = 1 \right\}$$

with $Q = Q(\varepsilon, N)$ from (3.3). Note that $0 < Q < 1$ implies that $S_Q$ is nonempty.
To prove this claim, let \( \{ \tilde{h}_i^N \} \) be the solution to (3.1) computed on the mesh \( \{ x_i \} \) with mesh-widths \( h_1, h_2, \ldots, h_N \) and let \( \{ \tilde{h}_i \} \) be the mesh after regridding. Then, by (2.4), we have

\[
\int_{x_{i-1}}^{x_i} \tilde{M}^N(x) \, dx = \frac{1}{N} \int_0^1 \tilde{M}^N(x) \, dx, \quad i = 0, \ldots, N,
\]

while condition (3.3) implies that

\[
\int_{x_{i-1}}^{x_i} \tilde{M}^N(x) \, dx \leq \tilde{h}_i/Q \quad \text{and} \quad \frac{1}{N} \int_0^1 \tilde{M}^N(x) \, dx \geq 1/N.
\]

Hence we have \( \tilde{h}_i \geq Q/N \) for \( i = 1, \ldots, N \), i.e. indeed, \( \Phi \) maps \( S_Q \) into itself.

The nonempty set \( S_Q \) is convex and compact, and \( \Phi \) is continuous. By the Brouwer fixed-point theorem—see, e.g., Smart (1974)—\( \Phi \) has a fixed point in \( S \). That is, there exists a mesh on which the computed solution satisfies \( \int_{x_{i-1}}^{x_i} \tilde{M}^N(x) \, dx = \int_{x_{j-1}}^{x_j} \tilde{M}^N(x) \, dx \) for all \( i \) and \( j \).

\[\square\]

### 3.2 Existence of a solution for equidistribution problem (1.7) and quasi-equidistribution problem (2.5)

Now we shall apply the above general theorem to our particular (quasi-)equidistribution problems.

**Corollary 3.2** For each \( \varepsilon \) and each \( N \), the equidistribution problem (1.7), (1.8) and the quasi-equidistribution problem (2.5), (1.8) have a solution.

**Proof.** It suffices to show that the equidistribution problem (1.7), (1.8) has a solution. Note that, by (1.2), discrete problem (1.3) satisfies the hypotheses of Theorem 3.1. Next, we verify that the monitor function \( M_i^N \) from (1.8) applied to problem (1.3) satisfies hypothesis (3.3). Combining (1.8) with (1.3), yields

\[
M_i^N \leq \varepsilon^{-1} \max_{0 \leq s \leq N} \sqrt{b(x_i, u_i^N)} + 1.
\]

Now, recall the bound \( |u_i^N| \leq C := \gamma_0 \max_s |b(x, 0)| \) from (Kopteva, 2007, Lemma 3.1), which implies that \( M_i^N \leq Q^{-1} \) with

\[
Q := (\varepsilon^{-1} \max_{0 \leq s \leq 1} \sqrt{b(x, s)} + 1)^{-1}.
\]

Thus condition (3.3) is indeed satisfied. Furthermore, \( M_i^N \) for each \( i \) is continuous in \( \{ u_i^N \} \) and \( \{ x_i \} \) and bounded; therefore \( \int_0^1 M_i^N(t) \, dt \) is continuous in \( \{ u_i^N \} \) and \( \{ x_i \} \), where \( M_i^N(x) := M_i^N \) for \( x \in (x_i-1, x_i) \). Hence the assertion of the corollary follows from Theorem 3.1 applied to the equidistribution problem (1.7), (1.8).

**Corollary 3.3** For each \( \varepsilon \) and each \( N \), the equidistribution problem (1.7) and the quasi-equidistribution problem (2.5), in which \( M_i^N \) is replaced by \( \tilde{M}_i^N \) from (1.5), (1.4b), have a solution.

**Proof.** Simply imitate the proof of Corollary 3.2. \[\square\]

### 4. Maximum mesh size for meshes \( \{ x_i^{(k)} \} \) generated by the algorithm

In the present section we establish an important property of any mesh generated by the algorithm that \( \max_i h_i^{(k)} \leq CN^{-1} \) independently of \( k, \varepsilon \) and \( N \); see Corollary 4.3 below. Not only the final mesh generated by the algorithm enjoys this property, but so do all the intermediate meshes. Thus we are guaranteed to avoid what is sometimes referred to as mesh starvation, when the mesh nodes are drawn away from regions where the solution is smooth.

First, we give two auxiliary lemmas.
LEMMA 4.1 On an arbitrary mesh \( \{x_i\} \), let
\[
B_i = \prod_{j=1}^{i} \left( 1 + \alpha h_j / \epsilon \right)^{-2} \quad \text{and} \quad \tilde{B}_i := B_N / B_i \quad \text{for} \quad i = 0, \ldots, N,
\]
where \( \alpha := \gamma_1 / \sqrt{6} \). Then for \( L^N B_i := -\epsilon^2 \delta^2 B_i + p_i B_i \), where \( i = 1, \ldots, N - 1 \) and \( \gamma_0^2 \leq p_i \leq \gamma_1^2 \), we have \( L^N \tilde{B}_i \geq 0 \). Furthermore, \( L^N \tilde{B}_i \geq 0 \) and we also have
\[
\sqrt{B_i} h_i / \epsilon = (\sqrt{B_{i-1}} - \sqrt{B_i}) / \alpha, \quad \sqrt{B_{i-1}} h_i / \epsilon = (\sqrt{B_i} - \sqrt{B_{i-1}}) / \alpha, \quad (4.1)
\]
and
\[
\sqrt{B_i} \leq (1 + \alpha x_i / \epsilon)^{-1}. \quad (4.2)
\]

Proof. Relations (4.1) are verified by a straightforward calculation, while (4.2) immediately follows from \( \prod_{j=1}^{i} (1 + \alpha h_j / \epsilon) \geq 1 + \sum_{j=1}^{i} (\alpha h_j / \epsilon) \). Now it suffices to prove the bound \( L^N \tilde{B}_i \geq 0 \), as its analogue for \( \tilde{B}_i \) is obtained similarly. Using the notation \( H_i := h_i / \epsilon \), we get
\[
\frac{B_i - B_{i-1}}{H_i} = - (2\alpha + \alpha^2 H_i) B_i
\]
and, similarly,
\[
\frac{B_{i+1} - B_i}{H_{i+1}} = \frac{2\alpha + \alpha^2 H_{i+1}}{(1 + \alpha H_{i+1})^2} B_i = \left( -2\alpha + \alpha^2 H_{i+1} \frac{3 + 2\alpha H_{i+1}}{(1 + \alpha H_{i+1})^2} \right) B_i,
\]
while
\[
L^N B_i \geq - \frac{2}{H_i + H_{i+1}} \left[ H_{i+1} \, \theta(\alpha H_{i+1}) + H_i \right] B_i + \gamma_0^2 B_i \geq [\gamma_0^2 - 6\alpha^2] B_i.
\]
Hence
\[
L^N \tilde{B}_i \geq - \frac{2\alpha^2}{H_i + H_{i+1}} \left[ H_{i+1} \, \theta(\alpha H_{i+1}) + H_i \right] B_i + \gamma_0^2 B_i \geq [\gamma_0^2 - 6\alpha^2] B_i,
\]
where \( \theta(t) := (3 + 2t) / (1 + t)^2 \leq 3 \) for \( t > 0 \). Now \( \alpha = \gamma_1 / \sqrt{6} \) implies \( L^N \tilde{B}_i \geq 0 \).

LEMMA 4.2 (BOUND ON TOTAL INTEGRAL OF MONITOR FUNCTION) Let \( \{\tilde{u}^N\} \) be the solution of (1.3) on an arbitrary mesh \( \{x_i\} \) such that \( \max_i h_i \leq 1 / 2 \), and let \( M^N \) be the monitor function from (1.8). Then for some \( C_i \geq 1 \) we have
\[
1 \leq I^N = \sum_{i=1}^{N} M^N_i h_i \leq C_1.
\]

Proof. Decompose the computed solution \( u^N \) into a smooth component \( w^N \) and a singular component \( v_N \) as follows; see (Kopteva, 2007, §6). Let \( u_0(x) \) be the unique solution of the reduced problem \( b(x, u_0) = 0 \). Next let \( w^N \) satisfy \( T^N w^N = 0 \) for \( i = 1, \ldots, N - 1 \), subject to \( w_0^N = u_0(0) \) and \( w_N^N = u_0(1) \). Then, by (Kopteva, 2007, Lemma 6.1), we get
\[
|\delta^2 w^N_i| \leq C, \quad i = 1, \ldots, N - 1.
\]
Now if \( u^N = w^N + v^N \), then for \( v^N \) we have
\[
L^N v^N_i = -\epsilon^2 \delta^2 v^N_i + p_i v^N_i = 0 \quad \text{for} \quad i = 1, \ldots, N - 1, \quad v^N_0 = -u_0(0), \quad v^N_N = -u_0(1), \quad (4.4)
\]
where \( p_i := \int_0^1 b_i(x, w_i^N + x v_i^N) \, ds \), which, by (1.2), satisfies \( \frac{\gamma}{\epsilon} \leq p_i \leq \frac{\gamma}{2} \). Hence

\[
1 \leq M_i^N \leq \min \{ \left| \delta^2 v_i^N \right|, \left| \bar{\delta}^2 v_i^N \right| \}^{1/2} + \max_i \left| \delta^2 w_i^N \right|^{1/2} + 1 \leq \frac{\eta}{\epsilon} \min \{ \left| v_i^N \right|, \left| \bar{v}_i^N \right| \}^{1/2} + C, \tag{4.5}
\]

where we used (4.3) and (4.4) (and, accommodating (2.2b), assumed that \( |v_i^N| \leq |v_i^N| \) and \( |\bar{v}_i^N| \leq |\bar{v}_i^N| \) only to simplify the presentation, as the other case poses no additional difficulties in the analysis).

Now, recalling Lemma 4.1 and applying the discrete comparison principle to problem (4.4), yields \( |v_i^N| \leq |u_0(0)|B_i + |u_0(1)|B_i \). Therefore,

\[
1 \leq I^N \leq C \left( \sum_{i=1}^{m} \frac{\sqrt{|B_i|} h_i}{\epsilon} + R_{m+1} + \sum_{i=m+2}^{N} \sqrt{|B_{i-1}|} h_i / \epsilon + 1 \right), \quad R_{m+1} := \min \left\{ \sqrt{B_m}, \sqrt{B_{m+1}} \right\} \frac{h_{m+1}}{\epsilon},
\]

where \( m \) it the unique integer such that \( B_i \geq B_i \) for \( i \leq m \) and \( B_i < B_i \) for \( i > m \). Next, invoking (4.1), we obtain \( I^N \leq C(1 + R_{m+1}) \), and to complete the proof, it suffices to show that \( R_{m+1} \leq C \). Note that \( h_{m+1} \leq 1/2 \) implies that either \( x_m \geq 1/4 \) or \( 1 - x_{m+1} \geq 1/4 \). Consider the case of \( x_m \geq 1/4 \), as the other case is similar. Then, by (4.2), we have \( R_{m+1} \leq \sqrt{B_m} h_{m+1} / \epsilon \leq \alpha^{-1} h_{m+1} / x_m \leq 2 \alpha^{-1} \). Thus indeed, for some constant \( C_1 \) we have \( I^N \leq C_1 \).

Next we give a basic property of meshes generated by the algorithm.

**COROLLARY 4.3** Let \( \{ x_i^{(k)} \} \) be any mesh generated by the algorithm. Then for all \( i \) we have

\[
h_i^{(k)} \leq C_i N^{-1}, \tag{4.6}
\]

**Proof.** Since \( C_1 \geq 1 \), inequality (4.6) clearly holds true when \( k = 0 \), so assume that \( k > 0 \). Combining (2.4) with \( 1 \leq M_i^{(k-1)} \), we get

\[
\int_{x_i^{(k-1)}}^{x_i^{(k)}} 1 \, dx \leq I^{(k-1)} / N.
\]

Hence \( h_i^{(k)} = x_i^{(k)} - x_i^{(k-1)} \leq I^{(k-1)} / N \leq C_1 / N \), by Lemma 4.2.

**COROLLARY 4.4** Suppose that the algorithm reaches its stopping criterion and halts. Let the final mesh and solution generated be \( \{ x_i^* \} \) and \( \{ u_i^* \} \), and \( M_i^* := M_i^{(k)} \) be the discrete monitor function on the final mesh. Then \( \max_i \{ M_i^* h_i^* \} \leq C_0 C_1 N^{-1} \).

**Proof.** The stopping criterion (2.3) implies that \( \max_i \{ M_i^* h_i^* \} \leq C_0 I^* / N \), where \( I^* \) is the total integral of the monitor function on the final mesh, for which, by Lemma 4.2, we have \( I^* \leq C_1 \). The desired result follows.

**REMARK 4.5** Note that the bound \( I^N \leq C_1 \) of Lemma 4.2 is essential for property (4.6) of the algorithm.

Consider alternative monitor functions \( M_i^N \) from the a posteriori error estimate (1.4) or \( M_i^N \) from (1.5). The corresponding total integrals of these monitor functions \( I^N := \sum_{i=1}^{N} M_i^N h_i \) and its analogue \( I^N \) might blow up in the sense that they are no longer bounded uniformly in \( \epsilon \) and \( N \). Indeed, e.g., on the uniform mesh we have \( M_i^N = O(\epsilon^{-1}) \) and hence \( I^{(k)} \geq M_i^{(k)} h_i = O(\epsilon^{-1}) N^{-1} \geq 1 \) if \( \epsilon \ll N^{-1} \). (This phenomenon is also reported in (Kopteva, 2007, §5).) Hence using \( M_i^N \) or \( M_i^N \) as a monitor function for our algorithm will result in so-called mesh starvation. If the contribution of \( |0, x_1| \) to \( I^N \) significantly exceeds the contribution of \( O(1) \) of the interior part of \( [0, 1] \), where the solution is smooth, step 4 of the algorithm will result in the mesh nodes being drawn away from the interior region; compare Figures 1 and 2 in §6. This is the reason for our choice of \( M_i^N \) instead of \( M_i^N \) or \( M_i^N \) in the algorithm.
5. How many iterations for \( \varepsilon \)-uniform second-order accuracy?

In this section we consider the linear case of (1.1):

\[
Lu := -\varepsilon^2 u'' + p(x)u = f(x), \quad u(0) = u(1) = 0, \tag{5.1}
\]

where, in agreement with (1.2), we have \( 0 < \gamma_0^2 \leq p(x) \leq \gamma^2 \). Now our discrete problem (1.3a) becomes

\[
L_N u_N^i = -\varepsilon^2 \delta^2 u_N^i + p(x_i)u_N^i = f(x_i) \quad \text{for} \quad i = 1, \ldots, N - 1, \quad u_N^0 = u_N^N = 0. \tag{5.2}
\]

Similarly to the decomposition of the computed solution \( u_N \) used in the argument of Lemma 4.2, the solution \( u \) of (5.1) can be decomposed into a smooth component \( w \) and a singular component \( v \) as \( u = w + v \), where \( Lw = 0 \) subject to the boundary conditions \( v(0) = -u_0(0) \) and \( v(1) = -u_0(1) \); compare with (4.4). In general, \( |v(x)| \leq |u_0(0)|e^{-\gamma_0|x|/\varepsilon} + |u_0(1)|e^{-\gamma|x-1|/\varepsilon} \), i.e. \( v(x) \) describes the boundary layers at \( x = 0 \) and \( x = 1 \). To focus on the mesh node distribution in one of the two boundary layer regions, we make the simplifying assumption that

\[
|u_0(0)| \geq C > 0, \quad u_0(1) = 0, \tag{5.3}
\]

which implies that \( u \) has a boundary layer at \( x = 0 \), but no boundary layer at \( x = 1 \). Note that if we had \( u_0(0) = u_0(1) = 0 \), then the solution of (5.1) would have no boundary layers and then, by (5.11) below, the computed solution would be second-order accurate on any mesh with \( \max_i h_i \leq CN^{-1} \).

Furthermore, for simplicity we assume that

\[
\varepsilon \leq N^{-1}, \tag{5.4}
\]

which is not a restriction in practical situations. Our analysis also works if, instead of (5.4), we have \( \varepsilon \leq \tilde{C}N^{-1} \) for some arbitrary but fixed constant \( \tilde{C} \).

We also assume that \( N \) is sufficiently large independently of \( \varepsilon \) and the iteration counter \( k \).

**Theorem 5.1** (Algorithm yields second-order accurate solution) Let \( u \) be the solution of (5.1) under assumptions (5.3), (5.4), and \( \{u_i^k\} \) be the \( k \)th solution computed by the algorithm on the \( k \)th mesh \( \{x_i^k\} \). Furthermore, let \( u^{(k)}(x) \) be the standard linear interpolant of \( \{u_i^k\} \). Then, if \( N \) is sufficiently large independently of \( \varepsilon \), there exists a positive integer \( K \), with \( K \leq C |\ln \varepsilon|/\ln N \), such that

\[
\max_{x \in [0,1]} |e^{(K)}(x)| \leq CN^{-2}, \quad \text{where} \quad e^{(k)} := u^{(k)}(x) - u(x) \quad \text{is the error in the} \ k \text{th solution.}
\]

The entire §5 is devoted to the proof of this theorem, which is divided into a series of intermediate results. To further help the reader, we group related material into subsections.

5.1 Notation and preliminary material

We introduce some notation that is used frequently later.

(i). Let \( t_1 \) and \( t_2 \) be arbitrary with \( 0 \leq t_1 < t_2 \leq 1 \). Recall that \( u_i^k \) is the computed solution on the \( k \)th mesh \( \{x_i^k\} \) generated by the algorithm and \( M_i^k(x) \) is the corresponding piecewise constant monitor function such that \( M_i^k(x) = M_i^k(x) \) for \( x \in (x_{i-1}^k,x_i^k) \); see §2, in particular (2.2). Define \( I^{(k)}[t_1,t_2] \) to be the integral of the monitor function \( M_i^k(x) \) over \( [t_1,t_2] \):

\[
I^{(k)}[t_1,t_2] = \int_{t_1}^{t_2} M_i^k(x) \, dx. \tag{5.5}
\]
We note that $I^{(k)}[0, 1] = I^{(k)}$ in the notation of §2.

(ii). Define the parameter $\sigma = \sigma(\varepsilon, N)$ by

$$
\sigma := \frac{2}{\gamma} \ln N \quad \text{so that} \quad e^{-\gamma \sigma / \varepsilon} = N^{-2},
$$

where $\gamma := \max_{x \in [0, \sigma]} \sqrt{p(x)} \geq \sigma_0$ with $\sigma := (2/\gamma_0) \varepsilon \ln N \geq \sigma$. This implies that

$$
\gamma - C \varepsilon \ln N \leq \sqrt{p(x)} \leq \gamma \quad \text{for} \quad x \in [0, \sigma].
$$

Note that the parameter $\sigma$ is essentially the transition point between the fine and coarse meshes in Shishkin meshes; see, e.g., Miller et al. (1996).

(iii). Applying the decomposition of the computed solution used in the argument of Lemma 4.2, to the $k$th computed solution, we get $u_i^{(k)} = w_i^{(k)} + v_i^{(k)}$. Here the smooth component $w_i^{(k)}$ satisfies $L^{(k)} w_i^{(k)} = f(x_i)$ on the mesh $\{x_i^{(k)}\}$ subject to $w_0^{(k)} = u_0(0)$ and $w_N^{(k)} = u_0(1)$, while the singular component $v_i^{(k)}$ satisfies (4.4) with $p_i = p(x_i)$ and, by (5.3), we have $v_N^{(k)} = 0$. So $\lambda_i^{(k)} = -u_0(0)z_i^{(k)}$ and $\lambda_i^{(k)} = w_i^{(k)} - u_0(0)z_i^{(k)}$, where $\{z_i^{(k)}\}$ is the solution to the problem

$$
L^{(k)} z_i^{(k)} = 0 \quad \text{on} \quad \{x_i^{(k)}\}_{i=1}^{N-1}, \quad z_0^{(k)} = 1, \quad z_N^{(k)} = 0.
$$

Note that we write $L^{(k)}$ for the difference operator $L^N$ on the $k$th mesh. Finally, we observe that since the discrete operator $L^{(k)}$ satisfies the maximum principle, for each $k$ the discrete function $z_i^{(k)}$ is decreasing in $i$.

(iv). For $i = 1, \ldots, N$ and $k = 0, 1, \ldots$, set

$$
\lambda_i^{(k)} = \lambda_i^{(k)}(N, i, k) = \frac{C_1}{C_3 \sqrt{z_i^{(k)} N}},
$$

where $C_1$ was defined in Lemma 4.2 and $C_3$ is defined below in Lemma 5.3. Many subsequent inequalities involve $\lambda_i^{(k)}$. It may help the reader to note that when $\lambda_i^{(k)}$ appears in our analysis, it usually turns out that $\lambda_i^{(k)} \leq C$ for some $C$.

Now we present an auxiliary a priori error estimate, which will be used below in the proof of Lemma 5.14.

**Lemma 5.2 (A Priori Error Estimate)** Let $u$ be a solution of (5.1) under assumptions (5.3), (5.4), and $\{u_i^N\}$ be a solution of (5.2) on an arbitrary mesh $\{x_i\}$ such that $h_i \leq CN^{-1}$ for some $C$. Furthermore, let $u^N(x)$ be the standard linear interpolant of $\{u_i^N\}$. Then

$$
|u^N(x) - u(x)| \leq C \max_{1 \leq i \leq N} \left[ \min \left\{ \frac{h_i}{\varepsilon}, 1 \right\} e^{-\gamma (1/2)(\varepsilon^{-1})} + N^{-1} \right]^2 \quad \text{for} \quad x \in [0, 1].
$$


Proof. First, note that for the solution \( u \) of (1.1) and the standard piecewise linear interpolant \( u^N(x) \) of (1.3) on an arbitrary mesh \( \{x_i\} \), we have the following \textit{a priori} error estimate:

\[
\max_i |u_i^N - u(x_i)| \leq C \left[ \max_{1 \leq i \leq N} \int_{x_{i-1}}^{x_i} ds \int_s^{x_i} e|u''(x)|dx + \max_{1 \leq i \leq N-1} \int_{x_{i-1}}^{x_i} ds \int_s^{x_i} \left| \frac{d^2}{dx^2} b(x, u(x)) \right| dx \right].
\]  

(5.11)

This is a version of (Kopteva, 2007, Lemma 3.2) and (Kopteva et al., 2005, Theorem 3.1).

Now, imitating the argument of (Kopteva et al., 2005, Corollary 3.1), in particular, representing \( u \) as in (Kopteva et al., 2005, (3.6), (3.8)), and then invoking \( f(1) = 0 \), which follows from (5.3), we get

\[
\left( \frac{d^m u}{dx^m} \right) \leq C(e^{-m} e^{-\sqrt{p(0)}\gamma/e} + \epsilon^2 - m + 1), \quad m = 0, \ldots, 3.
\]

Next, combining this with (5.11) and a standard linear-interpolation error bound, yields (5.10) with \( \gamma \) replaced by \( \sqrt{p(0)} \). Hence, to establish (5.10), it suffices to show that

\[
e^{-\sqrt{p(0)}\gamma/e} \leq 2[e^{-\gamma} + N^{-2}].
\]  

(5.12)

For \( x \leq \sigma \), by (5.7), (5.4), we get

\[
0 \leq [\gamma - \sqrt{p(0)}\gamma/e] \leq C\sigma^2 \epsilon / \epsilon \leq CN^{-1} \ln N,
\]

which follows from (5.3), we get

\[
\left( \frac{d^m u}{dx^m} \right) \leq C(e^{-m} e^{-\sqrt{p(0)}\gamma/e} + \epsilon^2 - m + 1), \quad m = 0, \ldots, 3.
\]

Next, combining this with (5.11) and a standard linear-interpolation error bound, yields (5.10) with \( \gamma \) replaced by \( \sqrt{p(0)} \). Hence, to establish (5.10), it suffices to show that

\[
e^{-\sqrt{p(0)}\gamma/e} \leq 2[e^{-\gamma} + N^{-2}].
\]  

(5.12)

For \( x \leq \sigma \), by (5.7), (5.4), we get

\[
0 \leq [\gamma - \sqrt{p(0)}\gamma/e] \leq C\sigma^2 \epsilon / \epsilon \leq CN^{-1} \ln N,
\]

and therefore (5.12). Furthermore, for \( x > \sigma \), combining \( e^{-\sqrt{p(0)}\gamma/e} \leq \epsilon^{\sqrt{p(0)}\sigma/e} \) with (5.13) and (5.6), again yields (5.12). Thus we established (5.12) for all \( x \in [0, 1] \). \( \square \)

5.2 Lower bounds on the discrete monitor function

Lemma 5.3 For \( i = 1, 2, \ldots, N \) and all \( k \), there exist positive constants \( C_2 \) and \( C_3 \) such that

\[
\sqrt{\zeta_i^{(k)}} \geq C_2 \epsilon \quad \text{for some} \quad i \quad \Rightarrow \quad \min_{j=1,\ldots,i} M_j^{(k)} \geq \frac{C_3 \sqrt{\zeta_i^{(k)}}}{\epsilon}.
\]

Proof. Since \( u_i^{(k)} = w_i^{(k)} - u_0(0)z_i^{(k)} \) (see §5.1(iii) for details), combining (2.2) with (4.3), we obtain

\[
M_i^{(k)} \geq |u_0(0)|^{1/2} \min \{ |\delta^2 z_i^{(k)}|, |\delta^2 z_i^{(k)}| \}^{1/2} - \max_i \delta^2 w_i^{(k)} |1/2 + 1 \geq |u_0(0)|^{1/2} \frac{\delta_0}{\epsilon} \sqrt{\zeta_i^{(k)}} - C'
\]

(see (4.5) for a similar upper bound on \( M_i^{(k)} \)). Here we also used \( e^2 \delta z_i^{(k)} \geq \delta^2 z_i^{(k)} \geq 0 \), which follows from (5.8), where \( z_i^{(k)} \) is decreasing in \( i \). Now for \( j \leq i \) we have

\[
M_j^{(k)} \geq \left( \delta_0 |u_0(0)|^{1/2} - \frac{C'}{C_2} \sqrt{\zeta_i^{(k)}} \right) \frac{\sqrt{\zeta_j^{(k)}}}{\epsilon}.
\]

Choosing \( C_2 \) sufficiently large, we get the assertion of the lemma. \( \square \)

The above lemma shows that, to get sharp lower bounds on our discrete monitor function \( M_i^{(k)} \), we need sharp lower bounds on \( z_i^{(k)} \). The next three lemmas provide such bounds in various situations.

\[
\sqrt{\zeta_i^{(k)}} \geq C_2 \epsilon \quad \text{for some} \quad i \quad \Rightarrow \quad \min_{j=1,\ldots,i} M_j^{(k)} \geq \frac{C_3 \sqrt{\zeta_i^{(k)}}}{\epsilon}.
\]
LEMMA 5.4 Let the sub-mesh \( \{x_i^{(k)}\}_{i=0}^{N'} \) be uniform for some integer \( N' > 1 \). Then

\[
z_i^{(k)} \geq B_0 - B_{N'} \quad \text{for} \quad i = 0, \ldots, N' , \quad \text{where} \quad B_i := \left( 1 + \gamma H_i^{(k)} / e \right)^{-2i}.
\]

**Proof.** Set \( H := h_i^{(k)} / e \). Now imitating the argument of Lemma 4.1, for \( i = 1, \ldots, N' - 1 \) we get

\[
L^{(k)} B_i \leq \frac{\eta^2}{H} \left( \frac{H \Theta(yH) + H}{\theta(yH) + H} \right) B_i + \gamma^2 B_i \leq \left[ \gamma^2 - \gamma^2 \right] B_i = 0
\]

(where \( \Theta(t) := (3 + 2t) / (1 + t)^2 > 0 \) for all \( t \geq 0 \), which implies \( L^{(k)} [B_i - B_{N'}] \leq 0 \). Furthermore, \( B_i - B_{N'} \leq 1 \) for \( i = 0 \) and \( B_i - B_{N'} = 0 \) for \( i = N' \). Since \( L^{(k)} z_i^{(k)} = 0 \), while \( z_0^{(k)} = 1 \) and \( z_N^{(k)} > 0 \), applying the discrete maximum/comparison principle, we get the assertion of the lemma. \( \square \)

LEMMA 5.5 Let the sub-mesh \( \{x_i^{(k)}\}_{i=0}^{N'} \) be uniform with \( x_{N'} = \sigma - C e \). Then for \( i = 0, \ldots, N \) we have \( z_i^{(k)} \geq e^{-\gamma_i^{(k)} / \epsilon} \).

**Proof.** Set \( H := h_i^{(k)} / e \). Then, by (5.7), for \( i < N' \) we get

\[
L^{(k)} e^{-\gamma_i^{(k)} / \epsilon} \leq \left[ \frac{-e^{\gamma H} - 1}{H^2} + 1 - e^{\gamma H} \right] e^{-\gamma_i^{(k)} / \epsilon} = \left[ - (\eta(yH / 2))^2 + 1 \right] e^{-\gamma_i^{(k)} / \epsilon} \leq 0,
\]

where \( \eta(t) := (\sinh t) / t \geq 1 \) for all \( t \). Next consider \( i \geq N' \) and note that \( e^2 \delta_i e^{-\gamma_i^{(k)} / \epsilon} = \gamma^2 e^{-\gamma_i^{(k)} / \epsilon} > 0 \) for some \( \xi_i \in [x_{i-1}^{(k)}, x_i^{(k)}] \). Combining this with (5.6), we obtain \( L^{(k)} e^{-\gamma_i^{(k)} / \epsilon} \leq \gamma^2 e^{-\gamma_i^{(k)} / \epsilon} \leq CN^{-2} \) for \( i \geq N' \). Thus there exists \( C \) such that \( L^{(k)} [e^{-\gamma_i^{(k)} / \epsilon} - CN^{-2}] \leq 0 \) for \( i = 1, \ldots, N - 1 \). Now, applying the discrete maximum/comparison principle yields the desired bound. \( \square \)

LEMMA 5.6 Suppose that for every \( i \) such that \( x_{i-1}^{(k)} \leq \sigma - C e \) we have \( h_i^{(k)} \leq C'' e N^{-1/2} \) for some constants \( C' \) and \( C'' \). Then \( z_i^{(k)} \geq e^{-\gamma_i^{(k)} / \epsilon} \).

**Proof.** Set \( \rho := 1 + CN^{-1/2} \) for some \( C \) that will be chosen below, and \( B(x) := e^{-\rho \gamma x / \epsilon} \). Then

\[
B(x) \geq e^{-\gamma x / \epsilon} \exp (-CN^{-1/2} \gamma \sigma / \epsilon) \geq e^{-\gamma x / \epsilon} / 2 \quad \text{for} \quad x \leq \sigma ,
\]

since, by (5.6), (5.4), for sufficiently large \( N \) we have \( CN^{-1/2} \gamma \sigma / \epsilon \leq CN^{-1/2} \ln N \leq \ln 2 \). Next, note that \( e^2 \delta_i^2 B(x_i^{(k)}) = e^2 B_i^2 (x_i^{(k)}) \) for some \( \xi_i \in [x_{i-1}^{(k)}, x_i^{(k)}] \) implies \( e^2 \delta_i^2 B(x_i^{(k)}) \geq \rho^2 \gamma^2 e^{-\rho \gamma x_i^{(k)} / \epsilon} \). Therefore, for \( x_i^{(k)} \leq \sigma - C e \), recalling that \( h_i^{(k)} \leq C'' e N^{-1/2} \), we get

\[
L^{(k)} B_i(x_i^{(k)}) \leq \left[ 1 - \rho^2 e^{-\gamma x_i^{(k)} / \epsilon} \right] \gamma^2 e^{-\gamma x_i^{(k)} / \epsilon} \leq 0,
\]

provided that \( C \) is sufficiently large so that \( \rho \geq \exp (CN^{-1/2} / 2) \). Furthermore, for \( x_i^{(k)} > \sigma - C e \), by (5.6), we get \( B(x_i^{(k)}) \leq C e^{-\rho \gamma x_i^{(k)} / \epsilon} \leq e^{-\gamma x_i^{(k)} / \epsilon} = CN^{-2} \), which implies \( L^{(k)} B(x_i^{(k)}) \leq \gamma^2 B_i(x_i^{(k)}) \leq CN^{-2} \); see the proof of Lemma 5.5 for a similar argument. Thus there exists \( C \) such that \( L^{(k)} B_i(x_i^{(k)}) - CN^{-2} \leq 0 \) for \( i = 1, \ldots, N - 1 \). Now, applying the discrete comparison principle yields \( z_i^{(k)} \geq B(x_i^{(k)}) - CN^{-2} \) for \( i = 0, \ldots, N \). Combining this with (5.14) implies the assertion of the lemma for \( x_i^{(k)} \leq \sigma \). For \( x_i^{(k)} > \sigma \), one again gets the assertion of the lemma with some \( C \geq 1/2 \) since \( z_i^{(k)} \geq 0 \), while, by (5.6), we have \( e^{-\gamma x_i^{(k)} / \epsilon} / 2 - (1/2)N^{-2} < 0 \). \( \square \)
5.3 Subdivision of $h_1^{(k)}$

The next lemma will show that if the first mesh interval is too coarse, then the next iteration of the algorithm will subdivide it $O(N)$ times.

**Lemma 5.7** Let $C_4$ be an arbitrary positive constant, $C_5 := 2(C_4^{-1} + \gamma)C_1/C_3$, and $N'$ be the greatest integer such that $N' \leq N/C_5$. Suppose that for some $k$, the mesh $\{x_i^{(k)}\}$ generated by the algorithm satisfies the condition of Lemma 5.4. Then

$$h_1^{(k)} \geq C_4 \varepsilon \quad \Rightarrow \quad h_1^{(k+1)} \leq C_5 h_1^{(k)}/N.$$

Furthermore, the mesh $\{x_i^{(k+1)}\}$ also satisfies the condition of Lemma 5.4 with the same $N'$.

**Proof.** By Lemma 5.4, setting $t := (1 + \gamma h_1^{(k)}/\varepsilon)^{-2}$, we get

$$z_1^{(k)} \geq t - t^{N'} = t(1 - t^{N'-1}) \geq t/4,$$

where $1 - t^{N'-1} \geq 1/4$ follows for sufficiently large $N$ from $t \leq (1 + \gamma C_4)^{-2}$. Thus we have

$$\sqrt{z_1^{(k)}} \geq \frac{1}{2(1 + \gamma h_1^{(k)}/\varepsilon)} \geq \frac{\varepsilon}{2h_1^{(k)}(C_4^{-1} + \gamma)}.$$

By (4.6), this implies $\sqrt{z_1^{(k)}} \geq \varepsilon/[2C_4N^{-1}(C_4^{-1} + \gamma)] \geq C_2 \varepsilon$ for sufficiently large $N$. Now, applying Lemma 5.3, we get $M_1^{(k)} \geq C_3 \sqrt{z_1^{(k)}/\varepsilon}$. Combining this with (5.5), (5.15) and Lemma 4.2, yields

$$f^{(k)}[0, x_1^{(k)}] = M_1^{(k)} h_1^{(k)} \geq \frac{C_3 \sqrt{z_1^{(k)}}}{\varepsilon} h_1^{(k)} \geq \frac{C_3 C_1}{2(C_4^{-1} + \gamma)} f^{(k)} = f^{(k)}/C_5$$

and furthermore,

$$f^{(k)}[0, x_1^{(k)}] \geq f^{(k)}/N.$$

Hence, by the construction of the algorithm, we have $x_1^{(k+1)} < x_1^{(k)}$ and

$$f^{(k)}[0, x_1^{(k+1)}] = f^{(k)}/N,$$

$$h_1^{(k+1)} = h_1^{(k)} \frac{f^{(k)}[0, x_1^{(k+1)}]}{f^{(k)}[0, x_1^{(k)}]}.$$

The second relation here follows from $f^{(k)}[0, x_1^{(k+1)}] = M_1^{(k)} h_1^{(k+1)}$ and $f^{(k)}[0, x_1^{(k)}] = M_1^{(k)} h_1^{(k)}$. Finally, a calculation yields the desired bound for $h_1^{(k+1)}$:

$$h_1^{(k+1)} = h_1^{(k)} \frac{f^{(k)}/N}{f^{(k)}[0, x_1^{(k)}]} \leq h_1^{(k)}(C_5/N).$$

Similarly, combining $f^{(k)}[0, x_1^{(k)}] \geq f^{(k)}/C_5$ with $f^{(k)}[x_j^{(k+1)}, x_j^{(k+1)}] = f^{(k)}/N$, for $j = 1, \ldots, N'$ we get $x_j^{(k+1)} \in [0, x_1^k]$ and $h_j^{(k+1)} = h_1^{(k+1)}$, i.e. the sub-mesh $\{x_j^{(k+1)}\}_{j=0}^{N'}$ is uniform and thus the next mesh $\{x_j^{(k+1)}\}_{j=0}^{N'}$ generated by the algorithm also satisfies the condition of Lemma 5.4 with the same value of $N'$. \hfill $\square$
To be more precise, under the conditions of Lemma 5.7, there exists an integer $N'' \geq N'$ such that
\[ h_j^{(k+1)} = h_i^{(k+1)}, \quad j = 1, \ldots, N''; \quad x_{N''}^{(k+1)} < h_1^{(k)} - h_1^{(k+1)}. \]

**Remark 5.8** To be more precise, under the conditions of Lemma 5.7, there exists an integer $N'' \geq N'$ such that
\[ h_j^{(k+1)} = h_i^{(k+1)}, \quad j = 1, \ldots, N''; \quad x_{N''}^{(k+1)} > h_1^{(k)} - h_1^{(k+1)}. \]

**Corollary 5.9** Let $C_4$ be an arbitrary positive constant. Then there exists a non-negative integer $K' = K'(\varepsilon, N, C_4)$ such that we have
\[ h_1^{(k)} \geq C_4 \varepsilon \quad \text{and} \quad h_1^{(k+1)} \leq C_3 h_1^{(k)} / N \quad \text{for} \quad k = 0, 1, \ldots, K' - 1; \quad h_1^{(K')} < C_4 \varepsilon, \]
where $C_3$ is defined in Lemma 5.7. Furthermore,
\[ K' \leq -\frac{\ln(C_4 C_3 \varepsilon)}{\ln(N/C_3)} \leq C \frac{\ln \varepsilon}{\ln N}. \]

**Proof.** Note that for $k = 0$ the mesh $\{x_i^{(k)}\}$ is uniform and hence satisfies the condition of Lemma 5.4 with $N'$ from Lemma 5.7. Now imitate the proof of (Kopteva & Stynes, 2001, Corollary 5.3). \[ \square \]

### 5.4 Properties of mesh intervals generated by the algorithm

This subsection contains two crucial corollaries that shed light on the lengths of certain mesh intervals generated inside the boundary layer. It may help the reader if we point out that a subscript $i$ usually corresponds to the mesh $\{x_i^{(k)}\}$, while a subscript $j$ usually corresponds to the mesh $\{x_j^{(k+1)}\}$.

**Lemma 5.10** Suppose that for some $i$ and $k$ we have $\sqrt{\varepsilon_i^{(k)}} \geq C_2 \varepsilon$.

(i) For any $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ such that $f^{(k)}[t_1, t_2] = f^{(k)} / N$, we have
\[ |t_2 - t_1| \leq \varepsilon \lambda_i^{(k)}. \] (5.16)

(ii) If
\[ x_i^{(k)} \geq \varepsilon \lambda_j^{(k)}, \] (5.17)
then
\[ f^{(k)}[0, x_i^{(k)}] \geq f^{(k)} / N. \] (5.18)

**Proof.** First note that, by (5.5) and Lemma 5.3, for $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ we get
\[ f^{(k)}[t_1, t_2] = \int_{t_1}^{t_2} M^{(k)}(x) \text{d}x \geq |t_2 - t_1| \frac{C_3 \sqrt{\varepsilon_i^{(k)}}}{\varepsilon}. \] (5.19)

(i) Suppose that $[t_1, t_2] \subseteq [0, x_i^{(k)}]$ and $f^{(k)}[t_1, t_2] = f^{(k)} / N$. Then, combining inequality (5.19), Lemma 4.2 and (5.9), immediately yields
\[ |t_2 - t_1| \leq \frac{\varepsilon f^{(k)}[t_1, t_2]}{C_3 \sqrt{\varepsilon_i^{(k)}}} = \frac{\varepsilon f^{(k)}}{C_3 \sqrt{\varepsilon_i^{(k)}} N} \leq \varepsilon \lambda_i^{(k)}. \]
Suppose that for some $\varepsilon > 0$ satisfies Condition 5.13, where $\varepsilon > 0$.

\[ C \]

\[ \text{Proof.} \] We imitate the proof of Corollary 5.1 in Kopteva & Stynes (2001). First, (5.18) implies that there is a unique point $x^* \in [0, x_i(1)]$ such that $I_k(x^*, x_i(1)) = I_k/N$. Hence (5.16) yields $x_i(1) - x^* \leq \varepsilon \lambda_i(1)$. But then $x_i(1) - \varepsilon \lambda_i(1)$ implies $x_i(1) - x^* \leq \varepsilon$, so $I_k(x_i(1), x_i(1)) = I_k/N$. By the construction of the algorithm, $I_k(x_i(1), x_i(1)) = I_k/N$. Consequently $x_i(1) \leq x_i(1)$. Now take $[t_1, t_2] = [x_i(1), x_i(1)]$ in (5.16) to get $h_i(1) = x_i(1) - x_i(1) - \varepsilon \lambda_i(1)$. 

\[ \text{COROLLARY 5.11} \] Suppose that for some $i$ and $k$, the inequalities $\sqrt{z_i(1)} \geq C_2 \varepsilon$ and $x_i(1) \geq \varepsilon \lambda_i(1)$ are satisfied. On the next mesh $\{x_i(1)\}$ generated by the algorithm,

\[ x_i(1) \leq x_i(1) - \varepsilon \lambda_i(1) \quad \text{for some } j \geq 1 \quad \Rightarrow \quad h_i(1) \leq \varepsilon \lambda_i(1). \]

\[ \text{Proof.} \] We imitate the proof of Corollary 5.2 in Kopteva & Stynes (2001). By the construction of the algorithm, $I_k(x_i(1), x_i(1)) = I_k/N$, while $x_i(1) \leq x_i(1)$ implies $[x_i(1), x_i(1)] \subseteq [0, x_i(1)]$. Thus we can take $[t_1, t_2] = [x_i(1), x_i(1)]$ in (5.16) to get $h_i(1) = x_i(1) - x_i(1) \leq \varepsilon \lambda_i(1)$. 

\[ \text{COROLLARY 5.12} \] Suppose that for some $i$ and $k$, the inequality $\sqrt{z_i(1)} \geq C_2 \varepsilon$ is satisfied. On the next mesh $\{x_i(1)\}$ generated by the algorithm,

\[ x_i(1) \leq x_i(1) \quad \text{for some } j \geq 1 \quad \Rightarrow \quad h_i(1) \leq \varepsilon \lambda_i(1). \]

\[ \text{Proof.} \] We imitate the proof of Corollary 5.1 in Kopteva & Stynes (2001). First, (5.18) implies that there is a unique point $x^* \in [0, x_i(1)]$ such that $I_k(x^*, x_i(1)) = I_k/N$. Hence (5.16) yields $x_i(1) - x^* \leq \varepsilon \lambda_i(1)$. But then $x_i(1) - \varepsilon \lambda_i(1)$ implies $x_i(1) - x^* \leq \varepsilon$, so $I_k(x_i(1), x_i(1)) = I_k/N$. By the construction of the algorithm, $I_k(x_i(1), x_i(1)) = I_k/N$. Consequently $x_i(1) \leq x_i(1)$. Now take $[t_1, t_2] = [x_i(1), x_i(1)]$ in (5.16) to get $h_i(1) = x_i(1) - x_i(1) \leq \varepsilon \lambda_i(1)$. 

\[ \text{5.5 A sufficient condition for second-order accuracy} \]

\[ \text{CONDITION 5.13} \] For every $i$ such that $x_i(1) \leq \sigma - C_6 \varepsilon$, let

\[ h_i(1) \leq C_7 \varepsilon. \]

\[ \text{LEMMA 5.14 (SUFFICIENT CONDITION FOR ACCURACY)} \] Suppose that for some $k \geq 0$ the mesh $\{x_i(1)\}$ satisfies Condition 5.13, where $C_6$ and $C_7$ are arbitrary constants such that

\[ C_6 \geq C_7 \geq C_1/(C_2 C_3), \quad e^{(C_6 - C_7)} \geq 2(C_2^2 + C_6), \]

for some constant $C_8$. Furthermore, let

\[ z_i(1) \geq e^{-\gamma_i(1)/2} - C_8 N^{-2}, \quad i = 0, \ldots, N. \]
(i) Then the mesh \{X_j^{(k+1)}\} also satisfies Condition 5.13 with \(C_6\) replaced by \(C_6 + C_7\).
(ii) Furthermore, there exists \(C\) such that
\[
|e^{(k+1)}(x)| = |u^{(k+1)}(x) - u(x)| \leq CN^{-2}, \quad x \in [0, 1],
\]  
where \(e^{(k+1)}(x) := u^{(k+1)}(x) - u(x)\) is the error in the computed solution on the \((k + 1)\)st mesh.
(iii) In fact, for any positive integer \(m\), the meshes \(\{X_j^{(k+m)}\}\) also satisfy Condition 5.13 with \(C_6\) replaced by \(C_6 + mC_7\), while the discrete functions \(\{X_j^{(k+m)}\}\) satisfy (5.22) with \(C_6\) replaced by some \(C_6(m)\), and we have \(|e^{(k+m)}(x)| \leq CN^{-2}\) for \(x \in [0, 1]\) and \(m = 1, 2, \ldots\), where \(C = C(m)\).

**Proof.** (i) Let \(n\) be the largest value of \(i\) for which \(X_{j-1}^{(k)} \leq \sigma - C_6\varepsilon\), so, by (5.20), we have
\[
x_{n-1}^{(k)} \leq \sigma - C_6\varepsilon < x_n^{(k)} \leq \sigma - (C_6 - C_7)\varepsilon.
\]  
To apply Corollary 5.11 with \(i = n\), we check its hypotheses. By (5.22), (5.24) and (5.6), one has
\[
x_n^{(k)} \geq \exp\left(\frac{-\gamma n^{(k)}}{\varepsilon}\right)/2 - C_6N^{-2} \geq \exp\left(-\gamma(\sigma - (C_6 - C_7)\varepsilon)/\varepsilon\right)/2 - C_6N^{-2}
\]  
\[\geq \left[\exp(C_6 - C_7)/2 - C_6\right]N^{-2}.
\]  
Thus, by (5.21) and (5.4), we get
\[
\sqrt{n^{(k)}} \geq C_2N^{-1} \geq C_2\varepsilon.
\]  
Now we verify the next hypothesis \(x_n^{(k)} \geq \varepsilon\lambda_n^{(k)}\) of Corollary 5.11. By (5.24), (5.6), we get
\[
x_n^{(k)} > \sigma - C_6\varepsilon = \varepsilon((2/\gamma)\ln N - C_6) \geq C_7\varepsilon
\]  
for \(N\) sufficiently large. But, by (5.9), (5.26) and (5.21), we have
\[
\varepsilon\lambda_n^{(k)} = \frac{C_1\varepsilon}{C_3 \sqrt{n^{(k)}} N} \leq \frac{C_1\varepsilon}{C_2\varepsilon} \leq C_7\varepsilon.
\]  
Thus the hypotheses of Corollary 5.11 with \(i = n\) are satisfied, while (5.27) implies \(\lambda_n^{(k)} \leq C_7\). Now Condition 5.13 for the mesh \(\{X_j^{(k+1)}\}\) with \(C_6\) replaced by \(C_6 + C_7\) follows from Corollary 5.11. Indeed,
\[
x_{j-1}^{(k+1)} \leq \sigma - (C_6 + C_7)\varepsilon = [\sigma - C_6\varepsilon] - C_7\varepsilon
\]  
implies, by (5.24) and (5.27), that
\[
x_{j-1}^{(k+1)} \leq x_{n-1}^{(k)} - \varepsilon\lambda_n^{(k)}.
\]  
Hence, by Corollary 5.11, we have
\[
h_{j}^{(k+1)} \leq C_7\varepsilon.
\]  
(ii) Consider two cases.

**Case A.** \(x_{j-1}^{(k+1)} \leq \sigma - (C_6 + C_7)\varepsilon\). Then (5.28) implies
\[
x_{j}^{(k+1)} \leq \sigma - C_6\varepsilon.
\]
Let $i$ be such that $x^{(k+1)}_j \in [x^{(k)}_{i-1}, x^{(k)}_i]$. Then $x^{(k)}_{i-1} \leq x^{(k+1)}_j \leq \sigma - C_6 \epsilon$, so $i \leq n$ by definition of $n$, while, by (5.20), we also have
\[ h^{(k)}_i \leq C_7 \epsilon. \tag{5.29} \]

Since $i \leq n$ and $z^{(k)}_i \geq z^{(k)}_n$, then, by (5.26), we can apply Corollary 5.12 to obtain
\[ h^{(k+1)}_j \leq C_1 \epsilon \cdot \frac{C_3 \sqrt{z^{(k)}_i} N}{C_3 \sqrt{z^{(k)}_i} N}, \]

which immediately yields
\[ \frac{h^{(k+1)}_j}{\epsilon} \exp\left(-\frac{\gamma^{(k+1)}_{j-1}}{2\epsilon}\right) \leq \frac{C_1}{C_3} \sqrt{\frac{z^{(k)}_i}{z^{(k)}_i} N} \exp\left(-\frac{\gamma^{(k+1)}_{j-1}}{2\epsilon}\right). \]

Examining (5.25) and (5.26), we note that $\exp(-\gamma^{(k)}_n/\epsilon)/2 - C_6 N^{-2} \geq C_2^2 N^{-2}$, which implies
\[ (C_2^2 + C_8) N^{-2} \leq \exp(-\gamma^{(k)}_n/\epsilon)/2 \leq \exp(-\gamma^{(k)}_n/\epsilon)/2. \]

Combining this with (5.22), we get
\[ z^{(k)}_i \geq \exp(-\gamma^{(k)}_n/\epsilon)/2 - C_6 N^{-2} \geq \frac{\exp(-\gamma^{(k)}_n/\epsilon)}{2(1 + C_6/C_2)}. \]

and hence
\[ \frac{h^{(k+1)}_j}{\epsilon} \exp\left(-\frac{\gamma^{(k+1)}_{j-1}}{2\epsilon}\right) \leq \frac{C_1}{C_3} \sqrt{\frac{z^{(k)}_i}{z^{(k)}_i} N} \exp\left(-\frac{\gamma^{(k+1)}_{j-1}}{2\epsilon}\right) \cdot \frac{\exp(-\gamma^{(k)}_n/\epsilon)}{2(1 + C_6/C_2)}. \tag{5.30} \]

Now (5.28) together with (5.29) implies that $x^{(k)}_i - x^{(k+1)}_{j-1} \leq 2C_7 \epsilon$. From (5.30) we therefore get
\[ \frac{h^{(k+1)}_j}{\epsilon} \exp\left(-\frac{\gamma^{(k+1)}_{j-1}}{2\epsilon}\right) \leq CN^{-1}. \tag{5.31} \]

Case B. $x^{(k+1)}_{j-1} > \sigma - (C_6 + C_7) \epsilon$. By (5.6), we have
\[ \exp\left(-\frac{\gamma^{(k+1)}_{j-1}}{2\epsilon}\right) = N^{-1} \exp\left(\gamma \cdot \frac{x^{(k+1)}_{j-1} - x^{(k+1)}_{j-1}}{2\epsilon}\right) \leq N^{-1} e^{(C_6 + C_7) \epsilon} \leq CN^{-1}. \tag{5.32} \]

Combining bounds (5.31) and (5.32) for cases A and B, for $j = 1, \ldots, N$ we get
\[ \min\left\{\frac{h^{(k+1)}_j}{\epsilon}, 1\right\} \exp\left(-\frac{\gamma^{(k+1)}_{j-1}}{2\epsilon}\right) \leq CN^{-1}. \tag{5.33} \]

Now the desired estimate (5.23) follows from Lemma 5.2 combined with Corollary 4.3.

(iii) Note that (5.33) also implies an analogue of the error estimate (5.23) for the solution $\{z^{(k+1)}_j\}$ of problem (5.8) with $k$ replaced by $k + 1$, and the solution $z(x)$ of the equation $Lz = 0$ subject to the
boundary conditions \( z(0) = 1 \) and \( z(1) = 0 \); thus we have \( |z_j^{(k+1)} - z(x_j^{(k+1)})| \leq CN^{-2} \). This yields an analogue of (5.22) on the mesh \( \{x_j^{(k+1)}\} \): 
\[
z_j^{(k+1)} \geq z(x_j^{(k+1)}) - CN^{-2} \geq e^{-\gamma t/2} - C'_{9}N^{-2}, \quad j = 0, \ldots, N,
\]
for some constant \( C'_{9} \). Here we used the bound \( z(x) \geq e^{-\gamma t/2} - CN^{-2} \), which is obtained imitating the argument of Lemma 5.5, but for the continuous operator \( L \).

Thus the hypotheses of the lemma (with \( C_6 \) and \( C_8 \) replaced by \( C'_{6} \) or \( C_7 \) and \( C'_{8} \), respectively, such that (5.21) holds true) apply on mesh \( k + 1 \), so by the same argument we obtain analogues of (i), (ii) and (5.22) on mesh \( k + 2 \). Continue repeating this argument a fixed number of iterations. \( \square \)

5.6 Proof of Theorem 5.1

To simplify the notation and presentation, it is convenient to prove the theorem with \( K \) replaced by \( K + 2 \). From Corollary 5.9 and (5.6), there exists \( K' \) such that
\[
h_1^{(K')} \geq \sigma, \quad h_1^{(K')} < \sigma, \quad h_1^{(K')} \leq C_2 h_1^{(K-1)}/N.
\]
Furthermore, by Remark 5.8 for Lemma 5.7, for some \( N'' > N/C_3 - 1 \) we have
\[
h_1^{(K)} = h_1^{(K)}, \quad i = 1, \ldots, N'', \quad x_i^{(K)} > h_1^{(K-1)} - h_1^{(K)}.
\] (5.34)

Let \( C_9 \) satisfy \( \ln((1 + \gamma C_9)/C_9) = \gamma/8 \); there exists such \( C_9 \) since the function \( \ln((1 + t)/t) \) equals 1 at \( t = 0 \) and is decaying to zero for \( t > 0 \), while \( (\gamma/8)/\gamma < 1 \). We now consider two cases.

Case A. \( h_1^{(K)} \leq C_9 \epsilon \). Combining this with \( h_1^{(K-1)} \geq \sigma \) and (5.34) implies \( x_i^{(K)} \geq \sigma - C_9 \epsilon \). Then, invoking Lemma 5.5, we get (5.22) for \( K = K' \) and some constant \( C_8 \). Furthermore, by (5.34), the mesh \( \{x_j^{(K)}\} \) satisfies Condition 5.13 with arbitrary \( C_6 \) and \( C_7 \) such that \( C_6 \geq 2C_9 \) and \( C_7 \geq C_9 \). Now choosing \( C_6 \) and \( C_7 \) that satisfy (5.21), and then invoking Lemma 5.14 yields the assertion of Theorem 5.1.

Case B. \( h_1^{(K)} > C_9 \epsilon \). First note that, by (5.6), the bound \( h_1^{(K)} \leq C_2 h_1^{(K-1)}/N \) implies \( h_1^{(K-1)} > (C_9/C_5) \epsilon N > 2\sigma \). Combining this with \( h_1^{(K)} < \sigma \) and (5.34), we observe that \( x_i^{(K)} > \sigma \). Now let \( n \leq N'' \) be such that \( \sigma \in [x^{(K)}_{n-1}, x^{(K)}_n] \). Then, by (5.6), we have
\[
n \leq \frac{\sigma}{h_1^{(K)}} + 1 \leq \frac{2\sigma}{h_1^{(K)}} = \frac{4 \ln N}{\gamma H}, \quad \text{where} \quad H := h_1^{(K)}/\epsilon > C_9.
\]
Next, Lemma 5.4 yields \( z_i^{(K)} \geq t^{1-n} - t^{-N''} \) with \( t := (1 + \gamma H)^{-2} \), where \( n \leq C \ln N \), while \( N'' > N/C_3 - 1 \). Then, imitating the argument used to obtain (5.15), for sufficiently large \( N \) we get
\[
\sqrt{\frac{1}{z_i^{(K)}}} \geq (1 + \gamma H)^{-n}/2 = \exp\left(-n \ln(1 + \gamma H)\right)/2 = \exp\left(-\frac{4 \ln N}{\gamma H} \ln(1 + \gamma H)\right)/2 \geq N^{-1/2}, \quad (5.35)
\]
where we used \( \ln((1 + \gamma H)/H) \leq \ln(1 + \gamma C_9)/C_9 = \gamma/8 \). Now, for sufficiently large \( N \), combining (5.35) with (5.4) yields \( \sqrt{z_i^{(K)}} \geq C_2N^{-1} \geq C_2 \epsilon \) and then, by (5.9),
\[
\lambda_{m}^{(K)} = \frac{C_1}{C_3 \sqrt{z_i^{(K)}}} N \leq \frac{2C_1}{C_3} N^{-1/2}.
\]
Hence, by Corollary 5.11 with \( i = n \), on the next mesh \( \{x_j^{(K+1)}\} \) generated by the algorithm we have
\[
x_{j-1}^{(K+1)} \leq \sigma - \varepsilon \frac{2C_1}{C_3}N^{-1/2} \text{ for some } j \geq 1 \quad \Rightarrow \quad h_j^{(K+1)} \leq \varepsilon \frac{2C_1}{C_3}N^{-1/2}.
\]
Therefore the mesh \( \{x_j^{(K+1)}\} \) satisfies Condition 5.13 for arbitrary \( C_6 \) and \( C_7 \); then choose \( C_6 \) and \( C_7 \) that satisfy hypothesis (5.21) of Lemma 5.14. Furthermore, by Lemma 5.6, the next hypothesis (5.22) is also satisfied for some \( C_8 \). Now invoking Lemma 5.14 yields the assertion of Theorem 5.1 in Case B.

6. Numerical results

First, we refer the reader to Kopteva et al. (2005), where extensive numerical results were presented for the algorithm of §2 that used a similar monitor function and was applied to both linear and nonlinear test problems. Mostly for completeness we shall give further numerical results here for one simple problem.

Our test problem is the linear equation from (5.1) with the boundary conditions
\[ u(0) = 2, \quad u(1) = -1, \]
in which \( p(x) = 4(1 + x)^{-4}[1 + \varepsilon(1 + x)] \), and whose exact solution
\[ u(x) = -\cos(2\pi t) + 3(e^{-1/\varepsilon} - e^{-1/\varepsilon})[1 - e^{-1/\varepsilon}]^{-1}, \quad t := 2x/(x + 1), \]
exhibits a boundary layer near \( x = 0 \); the right-hand side \( f \) and the boundary conditions are chosen so that (5.1) is satisfied; see Schatz & Wahlbin (1983), O’Riordan & Stynes (1986), Kopteva et al. (2005), Kopteva (2007).

Applying the algorithm to this test problem, instead of the monitor function \( M^N_i \), its scaled version \( \beta M^N_i \), from (2.6) was used with \( \beta := \max_i |u^N_i|^{1/2} = \sqrt{2} \); see Remark 2.4. (Similarly, we used the scaled version \( \bar{M}^N_{\beta i} \) of \( \bar{M}^N_{i} \), wherever appropriate, in our computations.)

Figure 1, which should be read from bottom to top, shows the mesh after each iteration. Each of these meshes is labeled with the value of \( C_0 \), for which the stopping criterion (2.3) becomes an equation; one can deduce how many iterations of the algorithm would have taken to converge given a priori a value of \( C_0 \). We shall take \( C_0 = 2 \) in the computations presented in Table 1. One can see from Figure 1 that

![Figure 1](image-url)
Grid equidistribution for a singularly perturbed reaction-diffusion problem

taking any smaller value of $C_0$ will increase the number of iterations without significantly changing the mesh. It should also be noted that the meshes computed by the algorithm are very close to a mesh by Bakhvalov (1969); see (Kopteva et al., 2005, Figure 5.4) for more details.

Figure 2 plots the meshes generated by a modification of the algorithm, in which $M^N_i$ in (2.2) was replaced by the scaled version $\bar{M}^N_i$ of $\bar{M}^N_i$ from the a posteriori error estimate (1.4). We observe a temporary mesh starvation: at the first few iterations all the mesh nodes are drawn into the narrow layer region at $x = 0$; compare with Figure 1 and also see Remarks 2.3 and 4.5 for a further discussion of this undesirable phenomenon.

Numerical results for the original version of the algorithm, as described in §2, are presented in Table 1. For various values of $\varepsilon$ and $N$, we computed the errors in the maximum norm, denoted $\|e\|_{\infty}$. Under each error is given the rate of convergence, expressed as a power of $N$ and computed in the standard way. These rates approach the value 2 as $N$ increases, confirming Theorem 5.1; the irregular

![Figure 2](image-url)

Fig. 2. Mesh movement when $M^N_i$ in (2.2) is replaced by $\bar{M}^N_i$ from (1.4); right-hand picture enlarges boundary layer; $\varepsilon = 10^{-5}$, $N = 32$. (Using $\bar{M}^N_i$ from (1.5) instead of $M^N_i$ yields similar results.)

Table 1. Accuracy and speed of algorithm: errors in the maximum norm $\|e\|_{\infty}$, their rates of convergence (even rows), number of iterations $K$, its ratio to the predicted number of iterations $O(|\ln \varepsilon|/(\ln N))$, and $C'$ to compare $M^N_i$ and $\bar{M}^N_i$.

| $N$  | $\varepsilon = 1$ | $\varepsilon = 10^{-3}$ | $\varepsilon = 10^{-8}$ | $\max_{\varepsilon=10^{-m}} \frac{|e|_\infty}{|e|_\infty}$ | $\max_{\varepsilon=10^{-m}} C'$ |
|------|-------------------|------------------------|------------------------|--------------------------------|------------------------|
| 32   | 1.44e-2 1         | 2.16e-2 3              | 2.83e-2 9              | 4.96e-2 1.96                  | 1.36                   |
|      | 1.96             |                        | 2.00                   | 2.54                          |                        |
| 64   | 3.71e-3 1         | 8.51e-3 3              | 7.08e-3 7              | 8.51e-3 1.81                  | 1.28                   |
|      | 2.00             |                        | 1.80                   | 1.23                          |                        |
| 128  | 9.29e-4 1         | 1.06e-3 2              | 2.03e-3 6              | 3.63e-3 1.28                  |                        |
|      | 1.98             |                        | 1.30                   | 2.14                          |                        |
| 256  | 2.35e-4 1         | 1.94e-4 2              | 8.25e-4 5              | 8.25e-4 1.31                  |                        |
|      | 2.04             |                        | 5.66                   | 2.19                          |                        |
| 512  | 5.74e-5 1         | 4.52e-5 2              | 6.54e-5 4              | 1.81e-4 1.26                  |                        |
|      | 2.02             |                        | 2.39                   | 2.96                          |                        |
| 1024 | 1.42e-5 1         | 1.11e-5 2              | 1.25e-5 4              | 2.33e-5 1.39                  |                        |
After the algorithm requires very few iterations, even for small \( \varepsilon \), and that our estimate \( O(\ln \varepsilon / (\ln N)) \) of the number of iterations is sharp.

Finally, note that our main results, Theorem 5.1, establishes that the algorithm achieves second-order accuracy after \( O((\ln \varepsilon / (\ln N)) \) iterations. But, strictly speaking, we have not proved and therefore are not guaranteed that after the algorithm reaches its stopping criterion, our computed solution is already second-order accurate. One can still estimate the error in terms of values obtained in the computation process invoking our a posteriori error estimate (1.4). Recall that (1.4) involves the monitor function \( M^N \) different from \( M^N_1 \), which we used in the algorithm. However, comparing \( M^N_1 \) and \( M^N \), intuitively one might expect that \( M^N \approx M^N_1 \) on a suitable layer-adapted mesh, which is indeed confirmed by numerical experiments. In particular, this is illustrated by the last column of Table 1, which describes \( C_1 = \max (M^N_1, b_1) / \max (M^N, b_1) \) on the final mesh produced by the algorithm for \( \varepsilon < 10^{-1} \) (for \( \varepsilon = 1 \) and \( \varepsilon = 10^{-1} \) we observe \( C_1 \leq 7.2 \) and \( C_1 \leq 2.9 \), respectively). We see from Table 1 that the values of \( C_1 \) are quite close to 1.3; the influence of \( C_1 \) on the accuracy is clarified in the following remark.

**Remark 6.1 (Accuracy of Solution When the Algorithm Terminates)** After the algorithm reaches its stopping criterion, compute \( C_1 = \max (M^N_1, b_1) / \max (M^N, b_1) \) (where \( M^N_1 := M^N_1 \) and \( M^N := M^N \) on the final mesh \( \{x_i^*\} \)); then, by Corollary 4.4 combined with the a posteriori error bound (1.4), we get

\[
\max_{0 \leq i \leq 1} |u^*(x) - u(x)| \leq C C_1^2 C_0^2 N^{-2},
\]

for the solution \( u \) of (1.1) and the linear interpolant \( u^*(x) \) of the final computed solution \( \{u_i^*\} \).

In summary, the numerical results obtained confirm our theoretical results and, furthermore, show that the monitor function \( M^N_1 \) used in the algorithm is superior to the other monitor functions considered in the sense that it guarantees any intermediate mesh generated by the algorithm from mesh starvation.

**References**


