Characterisation of Planar Maps

Consider a planar map $(x, y) \mapsto (f(x, y),\ g(x, y))$ with fixed point at $(x_e,\ y_e)$ which has the associated characteristic equation

$$\lambda^2 - a\lambda + b = 0$$

where $a = \left(\frac{\partial f}{\partial x}(x_e, y_e) + \frac{\partial g}{\partial y}(x_e, y_e)\right)$ and $b = \frac{\partial f}{\partial x}(x_e, y_e)\frac{\partial g}{\partial y}(x_e, y_e) - \frac{\partial f}{\partial y}(x_e, y_e)\frac{\partial g}{\partial x}(x_e, y_e)$.
(As usual a and b are the trace and determinant of the *Jacobian* matrix evaluated at the fixed point.)

The fixed point is classified as

stable node or focus	when	$\lvert b\rvert < 1$,	$\lvert a\rvert < \lvert 1+b\rvert$
completely unstable node or focus	when	$\lvert b\rvert > 1$,	$\lvert a\rvert < \lvert 1+b\rvert$
saddle	when		$\lvert a\rvert > \lvert 1+b\rvert$

Furthermore, the fixed point is a focus if $a^2 < 4b$, and a node or saddle otherwise.

Summarising (see Fig. 1 also)

Classification of $\mathbf{x}_e = \mathbf{0}$	Relationship between a and b
stable node (sink)	$\lvert a\rvert < \lvert 1+b\rvert,\ \lvert b\rvert < 1,\ (a/2)^2 > b$
unstable node (source)	$\lvert a\rvert < \lvert 1+b\rvert,\ \lvert b\rvert > 1,\ (a/2)^2 > b$
saddle	$\lvert a\rvert > \lvert 1+b\rvert$
stable focus (sink)	$\lvert a\rvert < \lvert 1+b\rvert,\ \lvert b\rvert < 1,\ (a/2)^2 < b$
unstable focus (source)	$\lvert a\rvert < \lvert 1+b\rvert,\ \lvert b\rvert > 1,\ (a/2)^2 < b$
centre	$\lvert a\rvert < 2,\ b = 1$

Note: $\lvert a\rvert = \lvert 1+b\rvert$ corresponds to the non-hyperbolic (degenerate) cases $\lvert\lambda_1\rvert = 1, \lvert\lambda_2\rvert = \lvert b\rvert$.

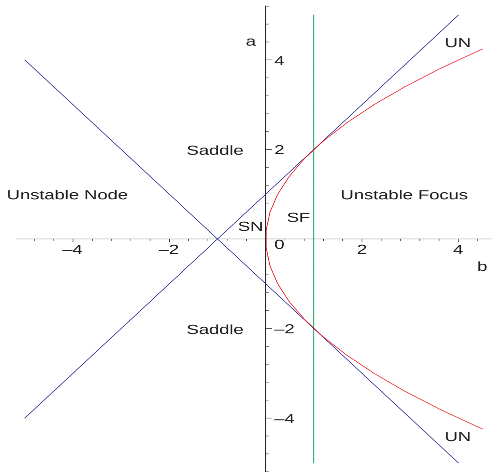

Figure 1: Stability in $a-b$ plane: SN = stable node, UN = unstable node, SF = stable focus