Bifurcation Theory

A *non-wandering set* of a dynamical system has the property that an orbit starting at any point of the set comes arbitrarily close arbitrarily often to the set. Examples of non-wandering sets are fixed points, limit cycles, invariant sets and chaotic orbits. Part of the remit of the subject of Dynamical Systems is to identify the non-wandering sets, determine whether they are stable or not, and whether they or their stability properties change as system parameter(s) change. This last phenomenon is called a *bifurcation*. Changes of stability and bifurcations go hand in hand.
For example, the 1-d flow

$$\dot{x} = px - x^2$$

with parameter p has fixed points $x_e[1] = 0$ and $x_e[2] = p$ with multipliers $Df(x_e[1]) = p$ and $Df(x_e[2]) = -p$ respectively. Thus $x_e[1] = 0$ is stable when $p < 0$ and unstable when $p > 0$. It undergoes a bifurcation at the critical value $p \triangleq p_c = 0$. Similarly $x_e[2] = p$ is unstable for $p < 0$ and stable for $p > 0$. It too undergoes a bifurcation at $p = p_c = 0$. These bifurcations can be tracked on a *bifurcation diagram*. [See Fig. 1 (Left)]

We distinguish between local and global bifurcations. The former can be analysed in terms of the changes in the local stability of invariant sets as parameters change, while the latter occur when larger invariant sets collide with each other or with fixed points or limit cycles. The term "co-dimension m" bifurcation is used when m parameters need to change to initiate the bifurcation. Some co-dimension 1 bifurcations follow.

Saddle-node (fold or tangent) bifurcation
The prototypical 1-d flow[1]

$$\dot{x} = p - x^2$$

has fixed points $x_e[1,2] = \pm\sqrt{p}$ with multipliers $\mp\sqrt{p}$ respectively. Thus there are no fixed points when $p < 0$ and two fixed points with opposing stability when $p > 0$. [See Fig. 1(right)]. $p_c = 0$ is the critical bifurcation value. The fixed points appear or disappear "out of the blue" as the parameter passes through the critical value.

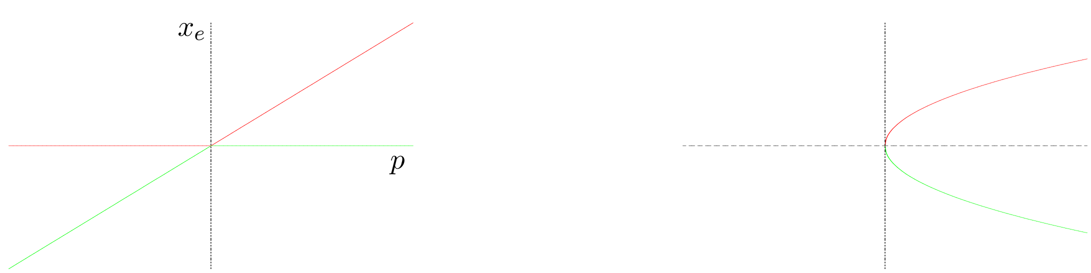

Figure 1: Bifurcation Diagrams: (left) Transcritical bifurcation and (right) Saddle-Node bifurcation. Red lines correspond to stable fixed points, green lines to unstable ones.

[1] Alternatively $\dot{x} = p + x^2$ is also used

Transcritical bifurcation
The prototypical 1-d flow[2] is

$$\dot{x} = px - x^2.$$

See the example above [Fig. 1(left)]. In summary, two fixed points which exist for all values of p and which exchange stability at $p = p_c$.

Pitchfork bifurcation
There are two types of pitchfork bifurcation:

1. The prototypical 1-d flow
$$\dot{x} = px - x^3$$
has fixed points $x_e[1] = 0,\ x_e[2,3] = \pm\sqrt{p}$ with multipliers $p,\ -2p,\ -2p$ respectively. There is one stable fixed point ($x_e[1] = 0$) for $p < 0$ which becomes unstable as p crosses through $p_c = 0$. In addition as p crosses through p_c, two additional fixed points ($x_e[2,3]$) are born. [see Fig. 2(left)]. This is called a *supercritical* pitchfork bifurcation.

2. The prototypical 1-d flow
$$\dot{x} = px + x^3$$
has fixed points $x_e[1] = 0,\ x_e[2,3] = \pm\sqrt{-p}$ with multipliers $p,\ -2p,\ -2p$ respectively. There is a stable fixed point ($x_e[1] = 0$) for $p < 0$ accompanied by two unstable fixed points ($x_e[2,3]$). The fixed point $x_e[1]$ becomes unstable as p crosses through $p_c = 0$, while the other two fixed points ($x_e[2,3]$) disappear as p crosses through p_c [see Fig. 2(right)]. This is called a *subcritical* pitchfork bifurcation.

Figure 2: Bifurcation Diagrams: (Left) Supercritical & (Right) Subcritical Pitchfork bifurcations

Bifurcations, as you might well suspect, are not a phenomenon of flows alone. They also occur in maps. Listed below are prototypical maps exhibiting the bifurcations discussed above, though not in the same order. Can you identify which is which and the

[2] Again $\dot{x} = px + x^2$ is also used

corresponding value of p_c?

$$\begin{aligned} x' &= (1+p)x - x^3 \\ x' &= p + x - x^2 \\ x' &= (1+p)x - x^2 \\ x' &= (1+p)x + x^3 \end{aligned}$$

Note that for flows the bifurcation occurring on the borderline between stability and instability naturally satisfies $Df(x_e, p_c) = 0$, i.e. the real part of the multiplier is zero. For the maps above the crossing through the borderline satisfies $Df(x_e, pc) = 1$, i.e. the modulus of the multiplier is one. There is however a second borderline through which maps can pass from stable to unstable behaviour or vice versa and that is when $Df(x_e, pc) = -1$ (the modulus of the multiplier is still one). This corresponds to a

Period-doubling (flip) bifurcation
The prototypical 1-d map[3]

$$x' = -(1+p)x + x^3$$

has fixed points $x_e[1] = 0$ and $x_e[2,3] = \pm\sqrt{2+p}$ (this latter pair of fixed points exists for $p > -2$ but will be ignored for the purposes of this discussion since they are unstable for all $-2 < p$ and hence cannot give rise to a bifurcation). $x_e[1]$ has multiplier $-(1+p)$ and hence is stable for $-2 < p < 0$ and unstable for $0 < p$. Thus $p_c = 0$. As p crosses from $p < p_c$ to $p > p_c$, the birth of a period-2 cycle $x_{1,2} = \pm\sqrt{p}$ occurs. [See Fig. 3]. Two points: firstly, period doubling bifurcations only occur for maps and secondly, a period doubling bifurcation for the map f is a pitchfork bifurcation for the map f^2. Hence the bifurcation discussed above can be classified as a supercritical flip bifurcation.

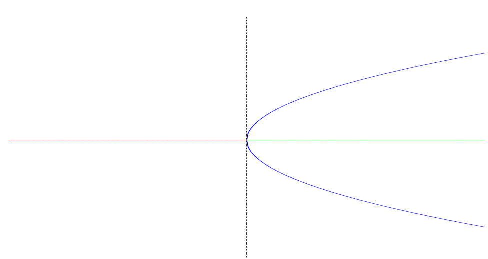

Figure 3: Bifurcation Diagram: Period-doubling bifurcation - blue lines correspond to stable 2-cycles

In addition to the bifurcations described so far, there is one frequently met co-dimension 1 bifurcation which occurs in 2-d and higher order systems, the *Hopf* bifurcation for flows (and its counterpart, the *Neimark-Sacker* bifurcation for maps). In each case the bifurcation occurs when a fixed point changes its stability with the concurrent birth of a limit cycle.

[3] And $x' = -(1+p)x - x^3$ is also used