1 Generalities: Flows & Maps

1. Linear Transformations
   The transformation $\mathbf{x} = T\mathbf{z}$ transforms

   (a) the linear flow $\dot{\mathbf{x}} = A\mathbf{x}$ to $\dot{\mathbf{z}} = T^{-1}AT\mathbf{z}$,

   (b) the linear map $\mathbf{x}' = A\mathbf{x}$ to $\mathbf{z}' = T^{-1}AT\mathbf{z}$,

   (c) the nonlinear flow

$$\begin{aligned} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) &= A\mathbf{x} + \mathrm{O}\big(\mathbf{x}\big) \\ \text{to} \qquad \dot{\mathbf{z}} = \mathrm{T}^{-1}\mathbf{f}(\mathrm{T}\mathbf{z}) &= T^{-1}AT\mathbf{z} + T^{-1}\,\mathrm{O}\big(T\mathbf{z}\big) \end{aligned}$$

   (d) and the nonlinear map

$$\begin{aligned} \mathbf{x}' = \mathbf{f}(\mathbf{x}) &= A\mathbf{x} + \mathrm{O}\big(\mathbf{x}\big) \\ \text{to} \qquad \mathbf{z}' = \mathrm{T}^{-1}\mathbf{f}(\mathrm{T}\mathbf{z}) &= T^{-1}AT\mathbf{z} + T^{-1}\,\mathrm{O}\big(T\mathbf{z}\big) \end{aligned}$$

   where $\mathrm{O}\big(.\big)$ are the nonlinear or higher order terms.
   In particular, if A is diagonalisable, then it is possible to choose

$$T = E \triangleq \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$$

   and

$$T^{-1}AT = \Lambda \triangleq \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\},$$

   where

$$A\mathbf{e}_i = \lambda_i \mathbf{e}_i, \qquad i = 1, 2, \ldots, n.$$

2. Computing A^k and e^{At} when A is diagonalisable ($A = E\Lambda E^{-1}$)

$$\begin{aligned} A^k &= E\Lambda^k E^{-1} \\ e^{At} &= Ee^{\Lambda t}E^{-1} \end{aligned}$$

3. The quadratic function

$$q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j = \mathbf{x}^T P \mathbf{x},$$

   where $P = P^T$, is positive definite iff *the leading principal minors* of P are all positive (*Sylvester*'s criterion).

4. The characteristic polynomial of the $n \times n$ matrix A is $\chi(\lambda) = det(\lambda I - A)$. It may be computed by

$$\chi(\lambda) = \lambda^2 - tr\,\lambda + det, \qquad \text{if } n = 2$$

$$\chi(\lambda) = \lambda^3 - tr\,\lambda^2 + \left(\sum_{i=1}^{3} M_{ii}\right)\lambda - det, \qquad \text{if } n = 3$$

where tr, M_{ii} and det are the trace, the minor associated with position (i, i) and the determinant of A respectively.

5. *Hartman-Grobman* Theorem: The behaviour of a dynamical system in a neighbourhood of a hyperbolic fixed point is qualitatively the same as that of the linearised system. A hyperbolic fixed point is one whose *Jacobian* matrix has **no** eigenvalue with

   (a) real part equal to zero (flows),

   (b) modulus equal to one (maps).

2 Flows

6. Stability of flows and characteristic polynomials:

   (a) A fixed point of the 2nd order flow with characteristic polynomial $\chi(\lambda) = \lambda^2 + a_1\lambda + a_0$ is stable iff

   $$a_1 > 0, \qquad a_0 > 0.$$

   (b) A fixed point of the 3-d flow with characteristic polynomial $\chi(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ is stable iff

   $$a_2 > 0,\ \ a_1 > 0,\ \ a_0 > 0, \qquad a_2 a_1 > a_0.$$

7. Invariant manifolds of 2nd order flows: If the flow

   $$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

   with fixed point at the origin has an associated invariant manifold $y = h(x)$, then h is the solution of

   $$g(x, h(x)) = Dh(x) f(x, h(x)), \qquad h(0) = 0$$

8. *Lyapunov*'s Direct Method for flows (*Lyapunov* Functions): For the system

   $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   the origin is

   (a) locally stable if there exists a positive definite function $V(\mathbf{x})$ with negative semi-definite time derivative $\dot{V}$,

   (b) locally asymptotically stable if $\dot{V}$ is negative definite (strict *Lyapunov* function).

The origin is globally asymptotically stable if in addition $V \to \infty$ as $||\mathbf{x}|| \to \infty$ (V is radially unbounded).

9. *Lyapunov*'s Direct Method for Linear flows: The origin of the system $\dot{\mathbf{x}} = A\mathbf{x}$ is globally asymptotically stable if for any real symmetric positive definite Q, the solution P of the *Lyapunov* Equation

$$A^T P + PA = -Q$$

is also real symmetric positive definite.

10. A modified *LaSalle* Invariance Principle for flows: For the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

if (i) V is positive definite, (ii) $\frac{dV}{dt}$ is negative semi-definite and (iii) the union of all invariant sets with $\frac{dV}{dt} = 0$ is $M = \{\mathbf{0}\}$, then $\mathbf{0}$ is asymptotically stable. In particular for the systems

$$\dot{x} = y, \qquad \dot{y} = -c(x) - b(y),$$

$$\dot{x} = y, \qquad \dot{y} = -c(x) - f(x)y,$$

where b and c are continuous functions with the same sign as their arguments and $f(x) > 0$, the origin $\mathbf{0}$ is asymptotically stable. This may be shown using

$$V(\mathbf{x}) = \frac{1}{2}y^2 + \int_0^x c(s)\, ds.$$

11. Index Theory:
    (a) The index of a node or focus is $+1$, of a saddle is -1.
    (b) Every periodic orbit encloses fixed points whose indices sum to $+1$.

12. *Dulac*'s criterion: Consider the 2nd order continuously differentiable flow $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$. If there exists a continuously differentiable function $\beta(x, y)$ such that on a simply connected region R_0,

$$D \triangleq \frac{\partial}{\partial x}\beta(x, y)f(x, y) + \frac{\partial}{\partial y}\beta(x, y)g(x, y)$$

is unchanged in sign, then there are no periodic orbits contained in R_0.

13. *Poincaré - Bendixson* theorem: Consider the 2nd order continuously differentiable flow $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$. If R_1 is a closed bounded trapping region containing no fixed points, then there exists a periodic orbit in R_1.

14. Cartesian-Polar transformation:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \arctan\left(\tfrac{y}{x}\right) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}, \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

15. The 1st order flow $\dot{x} = f(x, a)$ has a fixed point x_e which depends on the parameter a; x_e has a bifurcation at $a = a_c$ if $Df(x_e, a_c) = 0$.

16. *Hopf* Bifurcation: The flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a)$ has a fixed point $\mathbf{x}_e$ which depends on the parameter a. If the eigenvalues may be written as

$$\lambda(a) = \Re(a) \pm i\Im(a)$$

and for which, at $a = a_c$,

$$\Re(a_c) = 0, \qquad \Im(a_c) \neq 0, \qquad \frac{d\Re}{da}(a_c) \neq 0$$

then $\mathbf{x}_e$ has a *Hopf* Bifurcation at $a = a_c$.

3 Maps

17. Stability of 2nd order maps and characteristic polynomials: A fixed point of the 2nd order map with characteristic polynomial $\chi(\lambda) = \lambda^2 + a_1\lambda + a_0$ is stable iff

$$|a_1| < 1 + a_0, \qquad |a_0| < 1.$$

18. Invariant manifolds of 2nd order maps: If the map

$$x' = f(x, y), \qquad y' = g(x, y)$$

with fixed point at the origin has an associated invariant manifold $y = h(x)$, then h is the solution of

$$g(x, h(x)) = h\left(f(x, h(x))\right), \qquad h(0) = 0$$

19. *Lyapunov*'s Direct Method for maps (*Lyapunov* Functions): For the system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

the origin is

(a) locally stable if there exists a positive definite function $V(\mathbf{x})$ with negative semi-definite $\Delta V \triangleq V(\mathbf{x}') - V(\mathbf{x})$,

(b) locally asymptotically stable if ΔV is negative definite (strict *Lyapunov* function).

The origin is globally asymptotically stable if in addition $V \to \infty$ as $||\mathbf{x}|| \to \infty$ (V is radially unbounded).

20. *Lyapunov*'s Direct Method for Linear maps: The origin of the system $\mathbf{x}' = A\mathbf{x}$ is globally asymptotically stable if for any real symmetric positive definite Q, the solution P of the Discrete *Lyapunov* Equation

$$A^T PA - P = -Q$$

is also real symmetric positive definite.

21. The natural numbers are "*Sharkovsky*-ordered" by

$$3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd \ldots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^2 \cdot 7 \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

Sharkovsky's theorem says that if a 1st order map has a cycle of period p, then it also has cycles of all periods which appear to the right of p in the ordering.

22. The 1st order map $x' = f(x, a)$ has a fixed point x_e which depends on the parameter a; x_e has a bifurcation at $a = a_c$ if $|Df(x_e, a_c)| = 1$.

23. *Neimark-Sacker* Bifurcation: The map $\mathbf{x}' = \mathbf{f}(\mathbf{x}, a)$ has a fixed point $\mathbf{x}_e$ which depends on the parameter a. If the eigenvalues may be written as

$$\lambda(a) = R(a)e^{\pm i\theta(a)}$$

and for which, at $a = a_c$,

$$R(a_c) = 1, \qquad 0 < \theta(a_c) < \pi, \qquad \frac{dR}{da}(a_c) \neq 0, \qquad e^{\pm ik\theta(a_c)} \neq 1, \ \ k = 1, 2, 3, 4,$$

then $\mathbf{x}_e$ has a *Neimark-Sacker* Bifurcation at $a = a_c$.

24. *OGY* control: The chaotic map $\mathbf{x}' = \mathbf{f}(\mathbf{x}, a)$ has an unstable fixed point $\mathbf{x}_e$ which is embedded in an ergodic attractor at the nominal parameter value $a = a_0$. The fixed point may be stabilised by use of the strategy

$$a = \begin{cases} a_0 + K(\mathbf{x} - \mathbf{x}_e), & \text{if } ||\mathbf{x} - \mathbf{x}_e|| \leq \epsilon, \\ a_0, & \text{otherwise} \end{cases}$$

where K is chosen so that all the eigenvalues of $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_e, a_0) + KD_a\mathbf{f}(\mathbf{x}_e, a_0)$ have modulus less than 1, and where ϵ is the width of a neighbourhood (ball) around $\mathbf{x}_e$ such that $||K||\epsilon = \Delta a$, the maximum allowable parameter variation.