<div align="center">**Game Theory**</div>

Game Theory (GT) is the study of strategic interdependence. The typical "game" consists of players, actions, strategies and payoffs. The standard modes of analysis are once-played games in either

(i) matrix or strategic form where players' choice of actions are made simultaneously,

<div align="center">or</div>

(ii) extensive form where choice of actions are made sequentially.

1 Analysis of (finite) Matrix Games

One of the best known games is

Prisoner's Dilemma (PD)

<div align="center">

		Player 2	
		Keep Quiet	Confess
Player 1	Keep Quiet	(-1,-1)	(-12,0)
	Confess	(0,-12)	(-8,-8)

</div>

The solution is ⟨ confess, confess ⟩. (See IESDS).
Why don't the players coordinate to get ⟨ Keep Quiet, Keep Quiet ⟩ ?

The exact payoffs are irrelevant, the game can also be represented by the order of players' preferences - most preferred (p1) to least preferred (p4) :

<div align="center">

		Player 2	
		Keep Quiet	Confess
Player 1	Keep Quiet	(p2,p2)	(p4,p1)
	Confess	(p1,p4)	(p3,p3)

</div>

Other examples of PD-like games are

- War with strategies Defend, Attack respectively.

- Arms Race with strategies Pass, Build respectively.

- Free Trade/ Protection with strategies No Tax, Tax respectively.

- Advertising with strategies No Ads, Ads respectively.

Deadlock is another game (success is to fail!):

	Player 2	
Player 1	Try	Fail
Try	(0,0)	(-1,1)
Fail	(1,-1)	(0,0)

Using preferences, we can consider the more general version of deadlock

	Player 2	
Player 1	Left	Right
Up	(p2,p2)	(p1,p4)
Down	(p4,p1)	(p3,p3)

The solution is ⟨ Up, Left ⟩. (See IESDS). Neither player gets hir first choice unless the other makes a mistake.

1.1 Strict Dominance

Strategy X strictly dominates strategy Y for a player if X gives a bigger (more preferred) payoff than Y no matter what the other players do. Players never rationally choose strictly dominated strategies.

Reduce the matrix by **Iterated Elimination of Strictly Dominated Strategies** (IESDS) - see PD and Deadlock above. The order of elimination is irrelevant. Another example is the Dance Club game:

	Boon docks	
Downtown	Salsa	Hip Hop
Salsa	(80,0)	(60,40)
Hip Hop	(40,60)	(40,0)

The solution is ⟨ Salsa, Hip Hop ⟩. Salsa dominates Hip Hop for Club Downtown, then Boonies choses Hip Hop.

Some more examples

	Player 2		
Player 1	Left	Centre	Right
Up	(13,3)	(1,4)	(7, 3)
Middle	(4,1)	(3,3)	(6,2)
Down	(-1,9)	(2,8)	(8,-1)

Denoting strict dominance by $>$, then in the order given C $>$ R, M $>$ D, C $>$ L, M $>$ U. Thus the solution is ⟨ Middle, Centre ⟩.

Cournot Duopoly game. Firm 1 can produce i units at a cost of €1 each. Similarly firm 2 can produce j units at a cost of €1 each. The units sell on the market at a price of €$[8 - 2(i + j)]^+$ each, where $[.]^+$ is the positive part of $[.]$ The payoff to firm 1 is the profit gained which is €$[8 - 2(i + j)]^+ i - i$. Similarly the payoff to firm 2 is €$[8 - 2(i + j)]^+ j - j$. The game matrix is

	Firm 2			
	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$	(0,0)	(0,5)	(0, 6)	(0,1)
$i=1$	(5,0)	(3,3)	(1,2)	(-1,-3)
$i=2$	(6,0)	(2,1)	(-2,-2)	(-2,-3)
$i=3$	(1,0)	(-3,-1)	(-3,-2)	(-3,-3)

Firm 1 labels the rows.

The solution is $\langle i = 1, j = 1 \rangle$. What is the sequence of eliminations that leads to this?

All routes lead to the same result (proof?):

	Player 2	
	Left	Right
Up	(1,-1)	(4,2)
Middle	(0,2)	(3,3)
Down	(-2,-2)	(2,-1)

Player 1 labels the rows.

The solution is $\langle$ Up, Right $\rangle$.

1.2 Weak Dominance

Strategy X weakly dominates strategy Y for a player if X gives at least as big a payoff as Y no matter what the other players do and there is one at least one X payoff than is strictly greater than the corresponding Y payoff.

Iterated Elimination of Weakly Dominated Strategies (IEWDS) is in general not a valid technique. Consider the game:

	Player 2	
	Left	Right
Up	(0,1)	(-4,2)
Middle	(0,3)	(3,3)
Down	(-2,2)	(3,-1)

Player 1 labels the rows.

If we proceed as before and denoting weak dominance by $\geq$, we might argue that M $\geq$ U, L $\geq$ R. The solution is then $\langle$ Middle, Left $\rangle$. Alternatively we might argue that M $\geq$ D, R $\geq$ L which leads to the solution $\langle$ Middle, Right $\rangle$.

1.3 Best Response & Nash Equilibrium

Stag Hunt(SH) - it requires cooperation to catch a stag!

	Player 2	
	Stag	Hare
Stag	$(3^*, 3^*)$	(0,2)
Hare	(2,0)	$(1^*, 1^*)$

Player 1 labels the rows.

For this game there is no SDS nor WDS.

A **Nash equilibrium** (NE) is a set of strategies, one for each player, from which there is no incentive for any one player to deviate if all the other players play these strategies, i.e. no player can gain by changing, also called a "No regrets" choice. The *Best Response* of a player to another player's choice of strategy is the strategy that gives the largest or best payoff. We'll denote this by placing an * beside the payoff, e.g. in the stag game above , "stag" with associated payoff 3 is the best response of player 1 to player 2 playing "stag". Hence if both parts of a payoff pair have asterisks beside them, this must be a pure strategy *Nash* equilibrium (PSNE) - a pair of strategies where both players are playing deterministic strategies as opposed to a mixed strategy *Nash* equilibrium (MSNE), where players are randomly mixing between the strategies available to them.

Thus in the stag game, the PSNE solutions are ⟨ stag, stag ⟩ and ⟨ hare, hare ⟩. Notice that without efficient coordination, either solution is possible.

Consider the "Good buddies prisoner's dilemma" game:

	Player 2	
	Keep Quiet	Confess
Keep Quiet	(p1,p1)	(p4,p2)
Confess	(p2,p4)	(p3,p3)

Player 1 labels the rows.

This is identical to the stag hunt game.

An alternative definition of a NE is "mutual best response". Some further examples:
Traffic Lights(TL)

	Car 2	
	Go	Stop
Go	(-5,-5)	(1,0)
Stop	(0,1)	(-1,-1)

Car 1 labels the rows.

The PSNE solutions are ⟨ go, stop ⟩ and ⟨ stop, go ⟩.

Generals, Armies & Battles. Each general commands 3 armies. No battle occurs if either general puts 0 armies in the field. Otherwise the general with more armies wins the day. With i and j standing for the number of armies of General 1 and General 2 respectively in the battlefield, one possible game matrix is

	General 2			
	$j = 0$	$j = 1$	$j = 2$	$j = 3$
$i = 0$	(0,0)	(0,0)	(0, 0)	(0,0)
$i = 1$	(0,0)	(0,0)	(-1,1)	(-1,1)
$i = 2$	(0,0)	(1,-1)	(0,0)	(-1,1)
$i = 3$	(0,0)	(1,-1)	(1,-1)	(0,0)

General 1 labels the rows.

What are the PSNE(s) ?

1.4 Dominance & NE

If IESDS results in a unique solution then it is a (unique) NE. [proof by appeal to "no regrets"]. IESDS does not remove any NE.

IEWDS may lose NE. It is necessary to check using e.g. Best Responses. If you have a choice eliminate SDS before WDS. Examples:

- The example of Section 1.2 has two NE, each obtained by a different sequence of IEWDS.

- Consider the game

		Player 2	
		Left	Right
Player 1	Up	(2,3)	(4,3)
	Down	(3,3)	(1,1)

Using IEWDS, L $\geq$ R, then D > U. Hence the solution is ⟨ Down, Left ⟩. But using Best Responses, another NE is ⟨ Up, Right ⟩.

- The game

		Player 2		
		Left	Centre	Right
Player 1	Up	(2,2)	(4,2)	(4, 3)
	Middle	(2,4)	(5,5)	(7,3)
	Down	(3,4)	(3,7)	(6,6)

Using IEWDS C $\geq$ L, then M > U, M > D and C> R. Hence the solution is ⟨ Middle, Centre ⟩. This is the only NE.

1.5 MSNE

Matching Pennies(MP) is an example of a game with no PSNE.

		Player 2	
		Heads	Tails
Player 1	Heads	(1,-1)	(-1,1)
	Tails	(-1,1)	(1,-1)

Other names for this game are

- Goalkeeeper v. Penalty Taker

- Offense v. Defense (American Football)

- Fastball v. Curveball (Baseball)

- Attack A or B v. Defend A or B

Nash proved that every finite [1] game has a NE.
This game has the MSNE $\langle$ 1/2 Heads + 1/2 Tails, 1/2 Heads + 1/2 Tails $\rangle$.

More generally, MSNEs can be found using the *Bishop-Cannings* theorem. Consider the following game:

		Player 2	
		Left	Right
Player 1	Up	(3,-3)	(-2,2)
	Down	(-1,1)	(0,0)

Again note that there is no PSNE. We'll use the notation $R_1\langle s_1, s_2 \rangle$ and $R_2\langle s_1, s_2 \rangle$ to stand for the payoffs to Player 1 and 2 respectively when Player 1 plays strategy s_1 and Player 2 strategy s_2.
In the mixed strategy game let p be the probability that Player 1 plays Up, and similarly q the probability that Player 2 plays Left. Then the payoff to player 2 by playing Left against Player 1's mixed strategy is

$$R_2\langle p\text{Up} + (1-p)\text{Down}, \text{Left} \rangle = -3p + 1(1-p)$$

Again the payoff to player 2 by playing Right against Player 1's mixed strategy is

$$R_2\langle p\text{Up} + (1-p)\text{Down}, \text{Right} \rangle = 2p + 0(1-p)$$

If the two payoff values are different then Player 2 has a definite preference between the two and so would not want to randomise. If for instance Player 2 were to choose Left, then Player 1 would abandon his mixed strategy and choose Up instead. Similarly if Player 2 were to choose Right instead, Player 1 would change from randomising to playing Down. In either case, the mixed strategies go out the window and no NE exists. To get a MSNE requires that Player 2 be indifferent between the two payoffs i.e. requires that the two payoffs be the same and since the the payoff at any mixed strategy is a linear combination of the pure strategy payoffs, this must also have the same value. Equating the two payoffs gives

$$
\begin{aligned}
-3p + 1(1-p) &= 2p + 0(1-p) \\
\Rightarrow p &= 1/6
\end{aligned}
$$

and the value of the payoff is

$$v_2 = R_2\left\langle \frac{1}{6}\text{Up} + \frac{5}{6}\text{Down}, \text{Left or Right} \right\rangle = \frac{1}{3}$$

Similarly the payoffs to Player 1 by playing Up (respectively Down) against Player 2's mixed strategy are

$$R_1\langle \text{Up}, q\text{Left} + (1-q)\text{Right} \rangle = 3q + (-2)(1-q)$$

$$R_1\langle \text{Down}, q\text{Left} + (1-q)\text{Right} \rangle = (-1)q + 0(1-q)$$

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[1] finite number of players, finite number of pure strategies

respectively. To be indifferent between the two requires

$$3q + (-2)(1-q) = (-1)q + 0(1-q)$$
$$\Rightarrow q = 1/3$$

and the value of the payoff is

$$v_1 = R_1 \left\langle \text{Up or Down}, \frac{1}{3}\text{Left} + \frac{2}{3}\text{Right} \right\rangle = -\frac{1}{3}$$

Exercise: Show that the stag hunt game has a MSNE at $\langle$ 1/2 stag + 1/2 hare, 1/2 stag + 1/2 hare $\rangle$.

We can have partial MSNE where (at least) one player has a pure strategy and (at least) one player has a mixed strategy.

Examples of games with MSNE:

- Chicken (aka Snowdrift) :

		Player 2	
		Continue / Stay in car	Swerve / Shovel
Player 1	Continue / Stay in car	(-10,-10)	(2,-2)
	Swerve / Shovel	(-2,2)	(0,0)

  PSNEs occur at $\langle$ Continue, Swerve $\rangle$ and at $\langle$ Swerve, Continue $\rangle$. There is also a MSNE at $\langle$ 1/5 continue + 4/5 swerve, 1/5 continue + 4/5 swerve $\rangle$. Show that $v_1 = v_2 = -2/5$.

- Battle of the Sexes:

		Her	
		Ballet	Fight
Him	Ballet	(1,2)	(-1,1)
	Fight	(-1,1)	(1,-1)

  PSNEs occur at $\langle$ Ballet, Ballet $\rangle$ and at $\langle$ Fight, Fight $\rangle$. Show that there is a MSNE at $\langle$ 1/3 ballet + 2/3 fight, 2/3 ballet + 1/3 fight $\rangle$ with $v_1 = 2/3 = v_2$. Compare and contast this with the payoffs of the PSNEs.

1.6   MSNE and Dominance

A SDS cannot be played with positive probability in a MSNE (otherwise a higher payoff can be obtained by not playing the SDS when the strategy say to play it).

1.7 Strict Dominance in Mixed Strategies

Consider the game:

		Player 2	
		Left	Right
	Up	(3,-1)	(-1,1)
Player 1	Middle	(0,0)	(0,0)
	Down	(-1,2)	(2,-1)

This game has no SDS dominated by a pure strategy nor any PSNE. If a mixture of two pure strategies dominates another, then that strategy is a SDS.
In the above game 1/2 Up + 1/2 Down > Middle. Remove Middle to get

		Player 2	
		Left	Right
	Up	(3,-1)	(-1,1)
Player 1	Down	(-1,2)	(2,-1)

Show that a MSNE exists at ⟨ (3/5)Up + (2/5)Down, (3/7)Left + (4/7)Right ⟩.

Another example:

		Player 2		
		Left	Centre	Right
	Up	(-3,6)	(9,1)	(0, 2)
Player 1	Middle	(3,-4)	(2,4)	(4,1)
	Down	(4,7)	(3,2)	(-3,2)

Using IESDS, we get (1/4)L + (3/4)C > R, D > M, L > C, D > U. Thus the solution is ⟨ Down, Left ⟩.

1.8 Atypical Matrix Games

Almost all matrix games have an odd number of solutions (Wilson 1971). Examples of non generic games follow. Weak dominance is usually the culprit.

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		Player 2	
		Left	Right
	Up	(1,1)	(0,0)
Player 1	Down	(0,0)	(0,0)

  There are two PSNEs.

-

| | Player 2 | |
	Left	Right
Up	(2,2)	(9,0)
Down	(2,3)	(5,-1)

Player 1 (row labels Up, Down)

Using IESDS, L > R. Player 1 can choose Up or Down as pure strategies or any mixture of the two. All strategies yield a payoff of 2: an infinite number of strategies. If tempted to use IEWDS, U ≥ D, leading to the solution ⟨ Up, Right ⟩. Caveat emptor!

- Example 2 of Section 1.4 had two PSNEs. It also has an infinite number of MSNEs at ⟨ Up, qLeft $+ (1 - q)$Right ⟩ for $q \leq 3/4$.

- *Selten*'s Horse:

| | Player 2 | |
	Left	Right
Up	(3,1)	(0,0)
Down	(2,2)	(2,2)

Player 1 (row labels Up, Down)

There are two PSNEs at ⟨ Up,Left ⟩ and ⟨ Down,Right ⟩ and an infinite number of MSNEs at ⟨ Down, qLeft $+ (1 - q)$Right ⟩ for $q \leq 2/3$.

- Take or Share - TV game show:

| | Player 2 | |
	Share	Take
Share	(4,4)	(0,8)
Take	(8,0)	(0,0)

Player 1 (row labels Share, Take)

There are three PSNEs at all but ⟨ Share, Share ⟩. If one player chooses Take, then the other is indifferent between Share and Take which leads to an infinity of MSNEs.