Kaldor **Business Cycle**

1 Introduction

Nicholas Kaldor (1940) introduced the following macroeconomic model in order to explain business cycles:

$$\begin{aligned} \dot{Y} &= \alpha(I - S) \\ \dot{K} &= \quad I - \delta K \end{aligned} \tag{1}$$

where Y is income, I investment, S savings, K capital, α an adjustment factor and δ the capital depreciation rate. I and S are functions of Y and K. In addition *Kaldor* assumed that

$$\begin{aligned} I_Y &> 0 \\ I_K &< 0 \\ S_Y &> 0 \\ S_K &> 0 \end{aligned}$$

where a subscripted variable refers to the appropriate partial derivative. [The $S_K > 0$ assumption has been disputed by many economists, but is not crucial to the subsequent argument, and we'll ignore it.]
He further assumed that there existed a fixed point (Y_e, K_e) such that

$$I_{YY} \begin{cases} > 0, & \text{for } Y < Y_e \\ < 0, & \text{for } Y > Y_e \end{cases}$$

and, at which fixed point,

$$I_Y > S_Y$$

In addition, we will take the I and S functions to be given by

$$\begin{aligned} I &= \sigma(Y) - \beta K \\ S &= \qquad\quad \gamma Y \end{aligned} \tag{2}$$

where $\sigma(Y)$ is a sigmoid - shaped function, and β and γ are constant parameters (γ is the fraction of income saved)(See Fig.(1)), giving us the specific model

$$\begin{aligned} \dot{Y} &= \alpha\left(\sigma(Y) - \beta K - \gamma Y\right) \\ \dot{K} &= \quad \sigma(Y) - (\beta + \delta)K \end{aligned} \tag{3}$$

The fixed point (Y_e, K_e) satisfies

$$\begin{aligned} \sigma(Y_e) - \beta K_e - \gamma Y_e &= 0 \\ \sigma(Y_e) - \beta K_e - \delta K_e &= 0 \end{aligned}$$

from which we get

$$K_e = \frac{\gamma}{\delta} Y_e \tag{4}$$

$$\sigma(Y_e) - c Y_e = 0 \tag{5}$$

where $c = \gamma(1 + \beta/\delta)$.

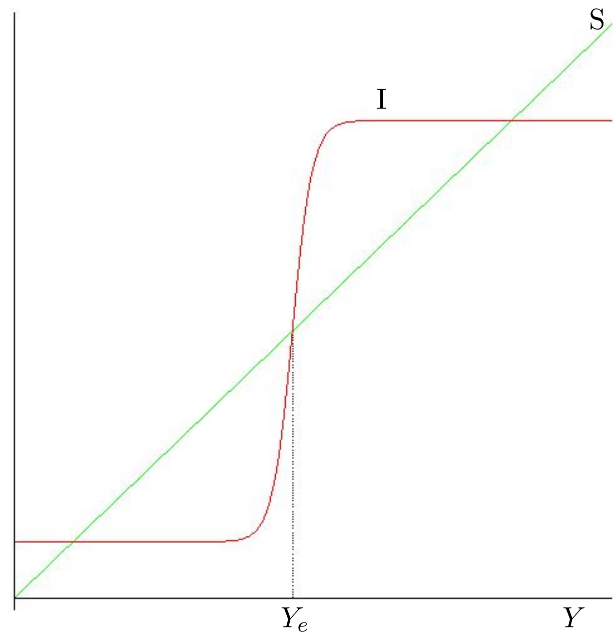

Figure 1: Investment $I(Y,.)$ and savings $S(Y,.)$ as functions of income Y for a fixed value of capital K.

2 Does the model exhibit cycles?

Figure (2) shows the nullclines of the flow:

$$K = N_1(Y) = \frac{\sigma(Y) - \gamma Y}{\beta}, \qquad (\text{at } \dot{Y} = 0) \tag{6}$$

$$K = N_2(Y) = \frac{\sigma(Y)}{\beta + \delta}, \qquad (\text{at } \dot{K} = 0) \tag{7}$$

in the case where $\sigma(Y) \geq \gamma Y$. It is straightforward to confirm that

(i) $\dot{Y} > 0$ below the red curve and $\dot{Y} < 0$ above it, while

(ii) $\dot{K} > 0$ below the green curve and $\dot{K} < 0$ above it.

Hence the tangent vectors are as indicated.
We choose Y_1 and K_1 such that the rectangle $\mathcal{R} = [0,\ Y_1] \times [0,\ K_1]$ is trapping where, for example we could choose

$$N_1(Y_1) < N_2(0)$$
$$K_1 = N_2(Y_1)$$

If the sigmoid function has minimum and maximum values σ_{min} and σ_{max} respectively then this gives

$$Y_1 \approx \frac{1}{\gamma}\left(\sigma_{max} - \frac{\beta}{\beta+\delta}\sigma_{min}\right), \qquad K_1 \approx \frac{\sigma_{max}}{\beta+\delta}$$

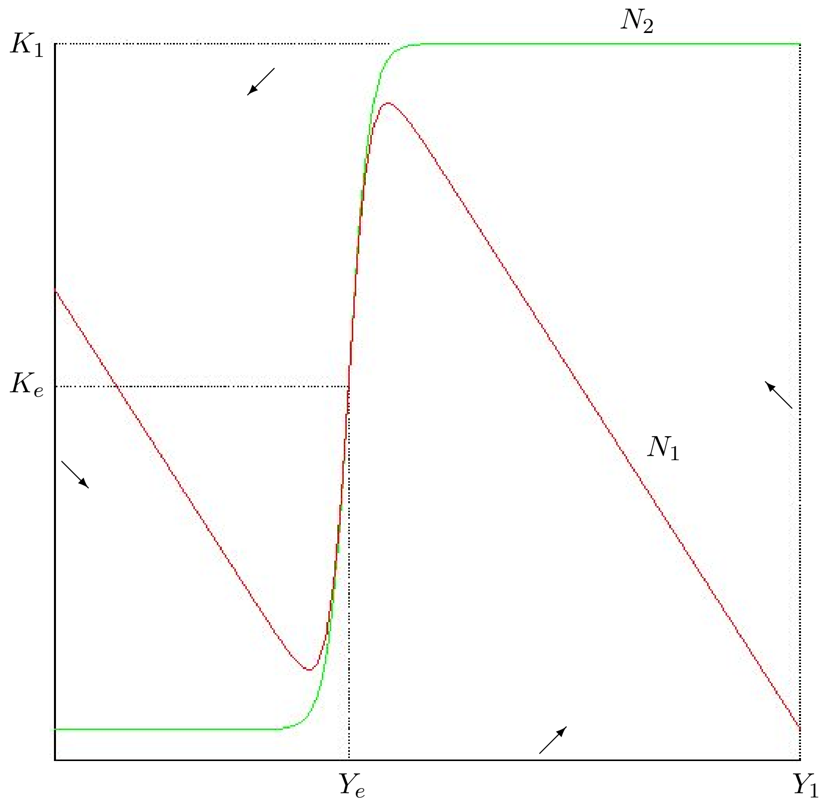

Figure 2: Nullclines in the $Y - K$ phase space.

The Jacobian matrix evaluated at (Y_e, K_e) is

$$J = \begin{bmatrix} \alpha(D\sigma(Y_e) - \gamma & -\alpha\beta \\ D\sigma(Y_e) & -(\beta+\delta) \end{bmatrix} \tag{8}$$

from which are obtained

$$tr = \alpha(D\sigma(Y_e) - \gamma) - (\beta + \delta), \qquad det = \alpha[\beta\gamma - \delta(D\sigma(Y_e) - \gamma)]$$

The characteristic equation is then $\lambda^2 - tr\lambda + det = 0$.

A repellor is a fixed point which is unstable but not a saddle. This corresponds to $tr > 0$ and $det > 0$. Hence the fixed point (Y_e, K_e) is a repellor if

$$\frac{\gamma\beta}{\delta} > D\sigma(Y_e) - \gamma > \frac{\beta+\delta}{\alpha} \tag{9}$$

and so is surrounded by a closed neighbourhood $(\mathcal{N})$ from which all trajectories exit. Thus a closed bounded trapping region $(\mathcal{R}\backslash\mathcal{N})$ exists in which there are no fixed points, and on which the flow is continuously differentiable. Hence the *Poincaré - Bendixson* theorem ensures that there is a limit cycle under the conditions given by (9).

3 *Hopf* bifurcation as α is varied

From the above discussion on the eigenvalues of the Jacobian, the fixed point (Y_e, K_e) is a focus if $tr^2 - 4det < 0$ i.e.

$$[\alpha(D\sigma(Y_e) - \gamma) - (\beta + \delta)]^2 - 4\alpha[\beta\gamma - \delta(D\sigma(Y_e) - \gamma)] < 0$$

Furthermore when $\alpha = \alpha_c$ defined by

$$\alpha_c(D\sigma(Y_e) - \gamma) - (\beta + \delta) = 0 \tag{10}$$

the real part (tr) of this focus is zero, the imaginary part ($\sqrt{\alpha[\beta\gamma - \delta(D\sigma(Y_e) - \gamma)]}$) is non-zero and since $\frac{d\,tr}{d\alpha}(\alpha_c) = D\sigma(Y_e) - \gamma \neq 0$ (by assumption), a *Hopf* bifurcation occurs at $\alpha = \alpha_c$. For $\alpha < \alpha_c$, (Y_e, K_e) is a stable focus, for $\alpha > \alpha_c$, (Y_e, K_e) is an unstable focus, and it may be shown that the bifurcation is supercritical.