Exercises on sequences, series and induction

1. Prove that the sum of the squares of the first n **odd** numbers is

$$\frac{n(4n^2-1)}{3}$$

2. The sequence $a_1,\ a_2,\ a_3, \ldots$ satisfies $a_1 = 15$, $a_{2011} = 57$ and for all $n \geq 3$, a_n is the arithmetic mean of the first $n-1$ terms. Find a_2.

3. Prove that for all $n \in \mathbb{N}$

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1) = \frac{n}{3}\left(4n^2 + 6n - 1\right).$$

4. If $a_1 = 1$ and

$$a_{n+1} = a_n + \frac{1}{2a_n},$$

prove that

$$44 < a_{2013} < 51$$

5. Find a formula for the sum of the first n terms of the series

$$1 - 3 + 5 - 7 + 9 - 11 + \cdots + (-1)^{n+1}(2n-1)$$

Prove it using mathematical induction. What is the value of the sum when $n = 2014$?

6. Prove that for all $n \in \mathbb{N}$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n+1}\frac{1}{n}$$

is positive.