Liénard Systems

The class of 2nd order systems given by

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

or equivalently by

$$\boxed{\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y} \tag{1}$$

is known to exhibit nonlinear oscillations if certain conditions are fulfilled.

1 Limit Cycle

If

- f and g are continuously differentiable,
- g is such that $xg(x) > 0, \quad x \neq 0$,
- f is an even function,
- the odd function $F(x) = \int_0^x f(s)\, ds$ has one positive zero at $x = a$ as shown in Fig (1), is positive and nondecreasing for $x > a$ and $F(x) \to \infty$ as $x \to \infty$.

then the *Liénard* system (1) has a unique stable limit cycle surrounding the origin.

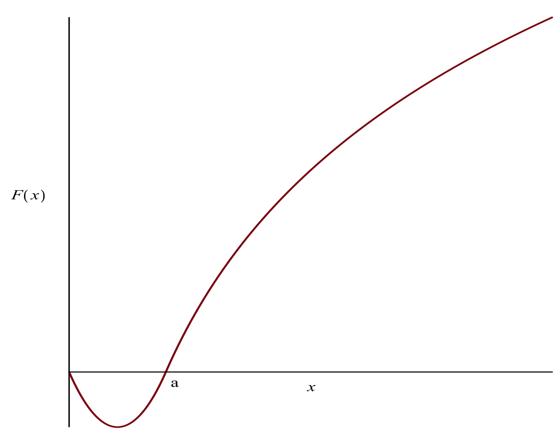

Figure 1: $F(x)$ v x

Example: The *van der Pol* oscillator is one of the best known of this class of oscillators:

$$\dot{x} = y, \qquad \dot{y} = -x - \mu(x^2 - 1)y, \qquad \mu \geq 0$$

Here $g(x) = x$, $f(x) = \mu(x^2 - 1)$ and

$$F(x) = \mu\left(\frac{1}{3}x^3 - x\right) = \frac{1}{3}\mu x(x^2 - 3)$$

Hence the positive zero occurs at $a = \sqrt{3}$. Thus the system has a stable limit cycle.

2 Stable fixed point

Consider the system (1). The origin is a stable fixed point if

- f and g are continuously differentiable,
- g is such that $xg(x) > 0, \qquad x \neq 0,$,
- $f(x) > 0$ in some neighbourhood of $x = 0$.

Consider the pd function

$$\begin{aligned} V(x, y) &= \frac{1}{2}y^2 + \int_0^x g(s)ds \\ \Rightarrow \frac{dV}{dt} &= y\dot{y} + g(x)\dot{x} \\ &= -f(x)y^2 - g(x)y + g(x)y \\ &= -f(x)y^2 \\ &\leq 0 \text{ for x near 0} \end{aligned}$$

Thus $\frac{dV}{dt}$ is nsd, implying that $\mathbf{x}_e = \mathbf{0}$ is stable. But is it asymptotically stable?
Now $\frac{dV}{dt} = 0$ only if $y = 0$ giving us that $R = \{(x, 0)^T\}$ - the x - axis. What is M? If $\mathbf{x} \in R$ but $x \neq 0$ then $\dot{y} = -g(x) \neq 0$ leading to the conclusion that y changes value i.e. is no longer 0 and so the trajectory leaves R. Hence if $\mathbf{x} \in M$ then x must be zero and so $M = \{(0, 0)^T\} = \mathbf{0}$. The Invariance Principle then tells us that $\mathbf{0}$ is locally asymptotically stable.
In addition, if (i) $f(x) > 0$ for all x and (ii) the integral $\int_0^x g(s)ds$ is unbounded as $|x| \to \infty$, then V is radially unbounded and we get that $\mathbf{0}$ is globally asymptotically stable.

Example: The "anti-*van der Pol* oscillator":

$$\dot{x} = y, \qquad \dot{y} = -x - \mu(x^2 - 1)y, \qquad \mu < 0$$

Here $g(x) = x$, $f(x) = \mu(x^2 - 1) > 0$ for $|x| < 1$. The origin is asymptotically stable but not globally asymptotically stable.