Control using *Lyapunov* Functions

In recent years, the use of *Lyapunov*'s Direct method has been investigated as a control design paradigm, in particular to stabilise systems. Assume that the control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{x}_e, \mathbf{0}) \tag{1}$$

has the fixed point $\mathbf{x}_e = \mathbf{0}$ corresponding to $\mathbf{u} \equiv \mathbf{0}$.[1]
Use of this approach to design control laws takes one of two courses

1 Specified Control Structure

Specify a structure for the control e.g. $\mathbf{u} = \kappa(\mathbf{x})$ where $\kappa : \mathbb{R}^n \to \mathbb{R}^r$ is a function from the state space to the control set which may contain unspecified parameters, after which the system becomes

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \kappa(\mathbf{x})) = \mathbf{F}(\mathbf{x})$$

The system is then analysed for stability by specifying or constructing a *Lyapunov* function. This in turn may constrain the parameter values needed to deliver asymptotic stability.
Example: consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1^2 u - x_2^5$$

With the control $u = -k_1 x_1 - k_2 x_2$ the closed-loop system becomes

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -k_1 x_1^3 - k_2 x_1^2 x_2 - x_2^5$$

Comparing the structure of this system with that of **Two canonical examples: 2-d flow** in the *Lyapunov*'s Direct Method notes, we see it is identical if we choose $k_1 > 0$, $k_2 = 0$, and thus by using $V(\mathbf{x}) = \frac{1}{2}x_2^2 + \int_0^{x_1} k_1 s^3 ds = \frac{1}{2}x_2^2 + \frac{1}{4}k_1 x_1^4$ we have a pd function. This in turn leads to $\frac{dV}{dt} = -x_2^6$ which is nsd, but the Invariance Principle arguments provide the basis for asymptotic stability. The radial unboundedness of $V(\mathbf{x})$ gives global asymptotic stability.

More generally, we can expand on this approach for 2nd order systems of the form e.g.

- A generalised spring-mass-damper system $\quad \ddot{x} + \mathrm{B}(\dot{x}) + \Gamma(x) = u$.
- A generalised *Liénard* system $\quad \ddot{x} + \Delta(x)\dot{x} + \Gamma(x) = u$.

In the first case, we attempt to choose $u = u(x, \dot{x})$ so that the resultant closed-loop system is of the form

$$\ddot{x} + b(\dot{x}) + c(x) = 0$$

where b and c are continuous functions with the same sign as their arguments i.e. " 1st & 3rd " functions (without justifying the terminology, let us call such a function "passive").

[1] We can similarly define fixed points corresponding to any fixed value of $\mathbf{u}$, after which we can transfer the fixed point to the origin using a change of variable.

In the second case we choose $u = u(x, \dot{x})$ so that the resultant closed-loop system is of the form

$$\ddot{x} + p(x)\dot{x} + c(x) = 0$$

where $p(x) > 0$ and c is a passive function.

Example: $\ddot{x} + \beta\dot{x}^2 - 2x = u, \; |\beta| < 2$. With $u = -F(\dot{x}) - G(x)$, the system equation becomes

$$\ddot{x} + \underbrace{\beta\dot{x}^2 + F(\dot{x})}_{b(\dot{x})} + \underbrace{G(x) - 2x}_{c(x)} = 0.$$

There is an obvious choice for the type of function that $G(x)$ should be to make $c(x)$ passive, e.g. $G(x) = 3x$. To make $b(\dot{x})$ passive, one possible choice is $F(\dot{x}) = \dot{x}^3 + k_1\dot{x}$ with k_1 chosen to make $\dot{x}^3 + \beta\dot{x}^2 + k_1\dot{x}$ an increasing function: this is achieved when $(2\beta)^2 - 4(3)k_1 < 0$. So, for instance, choose

$$k_1 = 2 > 4/3 \geq \beta^2/3.$$

The feedback controller is then $u = -\dot{x}^3 - 2\dot{x} - 3x$.

Example: $\ddot{x} + x^2\dot{x} + \gamma x^3 = u, \; -2 < \gamma < 1$. With $u = -F(x, \dot{x}) - G(x)$, the system equation becomes [2]

$$\ddot{x} + \underbrace{x^2\dot{x} + F(x, \dot{x})}_{p(x)\dot{x}} + \underbrace{G(x) + \gamma x^3}_{c(x)} = 0.$$

For $\epsilon > 0$, by choosing e.g. $F(x, \dot{x}) = \epsilon\dot{x}$ and $G(x) = (1 + \epsilon)x^3$,

$$p(x)\dot{x} = (x^2 + \epsilon)\dot{x}, \qquad c(x) = (1 + \epsilon + \gamma)x^3$$

both of which are passive. Hence the feedback controller is $u = -\epsilon\dot{x} - (1 + \epsilon)x^3$.

2 Control *Lyapunov* Function (clf)

Specify a candidate *Lyapunov* function $V(\mathbf{x})$ which is pd. Determine what constraints this function puts on the control when trying to ensure that $\frac{dV}{dt}$ is nd.
In respect to this approach, given a pd $V(\mathbf{x})$, then $\frac{dV}{dt} = \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(\mathbf{x}, \mathbf{u})$. Thus a sufficient condition for $\frac{dV}{dt}$ to be nd is that

$$\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(\mathbf{x}, \mathbf{u}) < 0, \qquad \text{for } \; \mathbf{x} \neq \mathbf{0} \tag{2}$$

Example (*Sontag* 1998): consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 + u$$

With $V(\mathbf{x}) = x_1^2 + x_2^2$, InEq (2) is

$$2x_1x_2 - 2x_2 \sin x_1 + 2x_2 u < 0, \qquad \mathbf{x} \neq \mathbf{0} \tag{3}$$

[2] In these systems it is sufficient to choose F so that the coefficient of the $\dot{x}$ term in $b(\dot{x})$ be positive.

Thus, when $x_2 \neq 0$, it is possible to choose u so that InEq (3) can be satisfied. However at any $\mathbf{x}$ for which $x_2 = 0$, it is impossible to satisfy InEq (3) since the left hand side (LHS) vanishes identically.
On the other hand if we use $V(\mathbf{x}) = 2x_1^2 + 2x_1x_2 + x_2^2$ which we can readily verify is pd, InEq (2) is

$$(4x_1 + 2x_2)x_2 + (2x_1 + 2x_2)(-\sin x_1 + u) < 0, \qquad \mathbf{x} \neq \mathbf{0} \tag{4}$$

Here, when $x_1 + x_2 \neq 0$, it is possible to choose u so that InEq (4) can be satisfied. And at any $\mathbf{x}$ for which $x_1 + x_2 = 0$, the LHS of InEq (4) becomes $-2x_1^2$. This is negative unless $x_1 = 0 \Rightarrow x_2 = 0$ and so InEq (4) is satisfied here also. Thus it is always possible to stabilise the system.
One possible feedback control [3] that stabilises the system is

$$u = \sin x_1 - k_1x_1 - k_2x_2$$

where $k_1 > 0$, $k_2 > 0$. Application of this control linearises the system, and it is straightforward to verify that the resultant system is (globally) asymptotically stable.

3 *Artstein*'s Theorem and *Sontag*'s Formula :

For the affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

the function $V(\mathbf{x})$ has

$$\frac{dV}{dt} = \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \dot{\mathbf{x}} = \underbrace{\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(\mathbf{x})}_{a(\mathbf{x})} + \underbrace{\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{g}(\mathbf{x})}_{b(\mathbf{x})}\,\mathbf{u}$$

Sontag defined $V(\mathbf{x})$ to be a clf if (i) it is pd and (ii)

$$\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{g}(\mathbf{x}) = \mathbf{0} \qquad \Rightarrow \qquad \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(\mathbf{x}) < 0, \qquad \mathbf{x} \neq \mathbf{0} \tag{5}$$

i.e. $\mathbf{u}$ can always be chosen so that $\frac{dV}{dt}$ is nd.

Artstein's Theorem (Artstein, 1983) states that a smooth (except possibly at the origin) stabilizer $\mathbf{u} = \kappa(\mathbf{x})$, $\kappa : \mathbb{R}^n \to \mathbb{R}^r$ can be constructed provided that a smooth (at least C^1) clf $V : \mathbb{R}^n \to \mathbb{R}$ can be found. *Sontag* gave a formula for choosing $\mathbf{u}$ in terms of the parameters of the clf. For a single input system, it is

$$u = \kappa(\mathbf{x}) \triangleq \begin{cases} -\frac{a(\mathbf{x}) + \sqrt{a^2(\mathbf{x}) + b^4(\mathbf{x})}}{b(\mathbf{x})}, & \text{if } b(\mathbf{x}) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

To verify that this works, we note that with this control

$$\frac{dV}{dt} = \left\{ \begin{array}{cc} -\sqrt{a^2(\mathbf{x}) + b^4(\mathbf{x})}, & \text{if } b(\mathbf{x}) \neq 0 \\ a(\mathbf{x}), & \text{otherwise} \end{array} \right\} < 0.$$

[3] A control strategy that cancels nonlinearities is usually to be avoided.

Example:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 2x_1x_2 + u$$

With the pd function $V(\mathbf{x}) = x_1^2 + x_1x_2 + x_2^2$, we calculate

$$\frac{dV}{dt} = \underbrace{\left(2x_1x_2 + x_2^2 + 2x_1^2x_2 + 4x_1x_2^2\right)}_{a(\mathbf{x})} + \underbrace{(x_1 + 2x_2)}_{b(\mathbf{x})} u$$

Then

$$b(\mathbf{x}) = 0 \qquad \Rightarrow \qquad \frac{dV}{dt} = -\frac{3}{8}x_1^2 < 0 \text{ for } \mathbf{x} \neq \mathbf{0}$$

and

$$u(\mathbf{x}) = \begin{cases} -\frac{(2x_1+x_2)x_2+2x_1x_2(x_1+2x_2)+\sqrt{[(2x_1+x_2)x_2+2x_1x_2(x_1+2x_2)]^2+[x_1+2x_2]^4}}{x_1+2x_2}, & \text{if } x_1 + 2x_2 \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

4 Cancellation Strategies

In Sections 1 & 2, examples of feedback controls which involved cancellation of nonlinearities were given. Such strategies are to be avoided unless compensated. There are two main reasons for this:

1. The mathematical model on which the control strategy is based may not be accurate. For example, consider the system with model equation $\dot{x} = x^3 + u$ to which the stabilising feedback control $u = -x^3 - 2x$ is applied. If the model is accurate, the closed-loop system becomes $\dot{x} = -2x$ which is stable. If, however, the true system equation is $\dot{x} = 1.08x^3 + u$, then the closed-loop system is $\dot{x} = 0.08x^3 - 2x$ which is unstable if $|x| > 5$.
   This can be compensated by a "robust" control law. Let's say we know that the system may be modelled by

   $$\dot{x} = \alpha x^3 + u, \qquad 0.9 < \alpha < 1.1,$$

   then the control law $u = -1.1x^3 - 2x$ results in the system equation

   $$\dot{x} + \underbrace{(1.1 - \alpha)x^3 + 2x}_{c(x)} = 0.$$

   Here $c(x)$ is passive and so the system is stable, irrespective of the true value of α.

2. Many times the nonlinearity may be beneficial! It can increase the speed of convergence to the fixed point ...at least when the state is far from equilibrium.

   Consider the system : $\dot{x} = -x^3 + u$. Contrast the two control strategies: (i) $u \equiv 0$ (blue) and (ii) $u = x^3 - x$ (red) [See Fig. 3]. In Fig 1, the system is initially at $x_0 = 10$; in Fig 2, the system is initially at $x_0 = 2$.

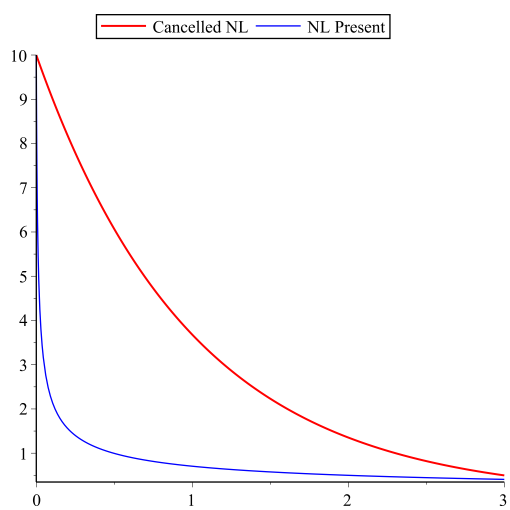

Figure 1: $x_0 = 10$

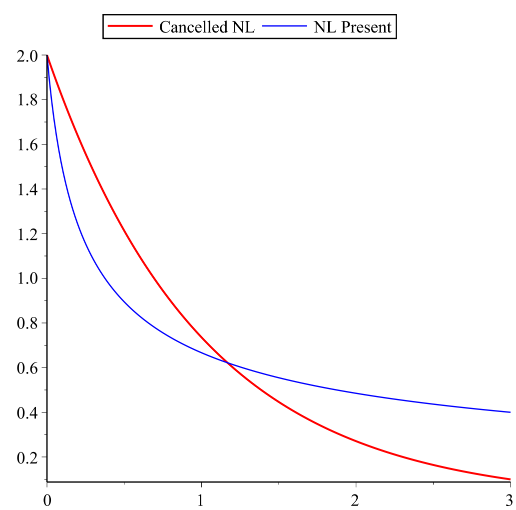

Figure 2: $x_0 = 2$

Figure 3: To cancel the nonlinearity or not?