Lyapunov's **Direct Method for Maps**

The method is essentially the same as for flows, the difference being that since we don't use time derivatives in discrete time, we need to replace $\frac{dV}{dt}$ in the development of the theory. In continuous-time, if $\frac{dV}{dt} \leq 0$ (respectively < 0) then the "energy of the system" is not increasing (respectively is decreasing). The quantity

$$\triangle V = V(\mathbf{x}') - V(\mathbf{x})$$

plays the same role in discrete-time, i.e. if $\triangle V \leq 0$ (respectively < 0) then the "energy of the system" is not increasing (respectively is decreasing). Hence we get

1 *Lyapunov* Local Stability Theorems

Consider the map

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}_e = \mathbf{0} \tag{1}$$

Define the *Lyapunov* function $V : \mathcal{N} \to \mathbb{R}$ by

(L1) V is pd

(L2) $\triangle V = V(\mathbf{x}') - V(\mathbf{x})$ is nsd

Theorem 1 (Stability) *If there exists a Lyapunov function for the system of Eq(1), then* $\mathbf{x}_e = \mathbf{0}$ *is stable.*

Theorem 2 (Asymptotic stability) *If there exists a Lyapunov function for the system of Eq(1), with the additional property that*
(L3) $\triangle V$ is nd
then $\mathbf{x}_e = \mathbf{0}$ *is asymptotically stable.*

Theorem 3 (Instability) *If there exists a pd function* V *for which* $\triangle V$ *is also pd for the system of Eq(1), then* $\mathbf{x}_e = \mathbf{0}$ *is unstable.*

2 *Lyapunov* Global Stability Theorems

These mirror the local stability theorems, but with some differences

- $\mathcal{N} = \mathbb{R}^n$ (of course),
- $V(\mathbf{x}) \to \infty$ as $||\mathbf{x}|| \to \infty$. A function with this property is said to be radially unbounded and it is needed to ensure that the contours of V define closed curves.

With these provisos, the global stability theorems read the same as the local stability ones.

3 Linear Systems

Even though *Lyapunov*'s first or indirect method (linearization) is usually adequate for linear systems, the use of the direct method gives us another tool for stability analysis. Recall that for linear systems, local stability and global stability are synonymous. For the linear system

$$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}_e = \mathbf{0} \tag{2}$$

consider the quadratic function

$$\begin{aligned} V(\mathbf{x}) &= \mathbf{x}^T P\mathbf{x}, \qquad P^T = P \\ \Rightarrow \triangle V &= (\mathbf{x}')^T P\mathbf{x}' - \mathbf{x}^T P\mathbf{x} \\ &= (A\mathbf{x})^T PA\mathbf{x} - \mathbf{x}^T P\mathbf{x} \\ &= \mathbf{x}^T A^T PA\mathbf{x} - \mathbf{x}^T P\mathbf{x} \\ &= \mathbf{x}^T (A^T PA - P)\mathbf{x} \\ &= -\mathbf{x}^T Q\mathbf{x} \end{aligned}$$

where we have set

$$\boxed{A^T PA - P = -Q} \tag{3}$$

This equation is known as the Discrete *Lyapunov* Equation. Note that Q is forced to be symmetric.
Let's consider the case where we want to establish asymptotic stability of the fixed point $\mathbf{x}_e = \mathbf{0}$. We require that (i) $V = \mathbf{x}^T P\mathbf{x}$ be pd or equivalently that the matrix P be pd and (ii) that $\triangle V = -\mathbf{x}^T Q\mathbf{x}$ be nd or equivalently that the matrix $-Q$ be nd, i.e. Q be pd. This leads to the following theorem

Theorem 4 (Linear Asymptotic Stability) *The (fixed point of the) system of Eq (2) is globally asymptotically stable if and only if, given any pd matrix Q, the solution P of Eq (3) is also pd.*

Example:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 3/8 & 1/4 \end{pmatrix} \mathbf{x}$$

Since Q can be any pd matrix, it is quite common to choose $Q = I$. Building in symmetry in P, the Discrete*Lyapunov* Equation for this system becomes

$$\begin{pmatrix} 0 & 3/8 \\ 1 & 1/4 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3/8 & 1/4 \end{pmatrix} - \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

or writing this as a system of linear equations

$$\begin{aligned} 9/64p_3 - p_1 &= -1 && \text{position } (1,1) \text{ in matrix equation} \\ -5/8p_2 + 3/32p_3 &= 0 && \text{position } (1,2) \text{ in matrix equation} \\ p_1 + 1/2p_2 - 15/16p_3 &= -1 && \text{position } (2,2) \text{ in matrix equation} \end{aligned}$$

(The equation in position $(2,1)$ is the same as that in position $(1,2)$, due to the built-in symmetry, and so there are only 3 independent equations in the 3 unknowns)

The system has solution

$$p_1 = \frac{107}{77}, \qquad p_2 = \frac{32}{77}, \qquad p_3 = \frac{640}{231}$$

So we have

$$P = \begin{pmatrix} 107/77 & 32/77 \\ 32/77 & 640/231 \end{pmatrix}$$

which has leading principal minors $P_{11} = \frac{107}{77}$ and $detP = \frac{9344}{2541}$. Thus $\mathbf{x}_e = \mathbf{0}$ is globally asymptotically stable.

4 *LaSalle*'s Invariance Principle

Consider the map

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}_0 \text{ given.} \tag{4}$$

LaSalle uses a slightly different definition for a *Lyapunov* function. We'll refer to it as a L-*Lyapunov* function:
Definition: Let $X \subset \mathbb{R}^n$. $V : X \to \mathbb{R}$ is a L-*Lyapunov* function if (i) V is continuous and (ii) $\triangle V \leq 0$.
If in addition, V is positive definite, then it coincides with the standard definition of a *Lyapunov* function.

Theorem 5 (Boundedness) *If there exists $K > 0$ such that V is L-Lyapunov function on the set $\{\mathbf{x} : ||\mathbf{x}|| > K\}$ and V is radially unbounded, then all solutions (orbits) of Eq(4) are bounded.*

Theorem 6 (Invariance Principle) *If V is L-Lyapunov function and the orbit of Eq (4) is bounded, then there exists c such that orbit approaches $M \cap V^{-1}(c)$ where M is the largest invariant set contained in $R = \{\mathbf{x} : \triangle V = 0\} \cap \overline{X}$.*

Corollary 7 (Asymptotic Stability) *If V satisfies the hypotheses of Theorem 6, and $M \cap V^{-1}(c) = \{\mathbf{0}\}$, then $\mathbf{0}$ is asymptotically stable.*

Example:

$$x' = \frac{ay}{1+x^2}, \qquad y' = \frac{bx}{1+y^2}, \qquad \mathbf{x}_e = \mathbf{0}$$

Consider

$$\begin{aligned} V(x,y) &= x^2 + y^2 \\ \Rightarrow \triangle V &= \left(\frac{b^2}{(1+y^2)^2} - 1\right) x^2 + \left(\frac{a^2}{(1+x^2)^2} - 1\right) y^2 \\ &\leq (b^2-1)x^2 + (a^2-1)y^2 \end{aligned}$$

If $a^2 \leq 1$ and $b^2 \leq 1$, then Theorem 5 applies and so all orbits of the system are bounded.

- If $a^2 < 1$ and $b^2 < 1$, then $\mathbf{0}$ is globally asymptotically stable.(Using Theorem 2 and the radial unboundedness of V.)

- If $a^2 < 1$ and $b^2 = 1$, then $\triangle V \leq (a^2 - 1)y^2$. Hence R is the x-axis, and $M = \{\mathbf{0}\}$. So $\mathbf{0}$ is globally asymptotically stable. (Using Corollary 7 and the radial unboundedness of V.)
- If $a^2 = 1$ and $b^2 = 1$, then $R = M$ equals the union of the x- and y- axes. By Theorem 6, all orbits tend to

  $$R \cap \{\mathbf{x} : V(\mathbf{x}) = c^2\} = \{(c, 0), (-c, 0), (0, c), (0, -c)\}.$$

  In the case where $ab = +1$, this set consists of two period 2 orbits, while in the case of $ab = -1$, the set consists of one period-4 orbit.