Ruling out Periodic Orbits

1 Index Theory

If C is a simple closed curve (i.e. no self intersections)-not necessarily a PO - that does not pass through any fixed points of the 2-d flow

$$\dot{x} = f(x,y), \qquad \dot{y} = g(x,y), \qquad \phi = \arctan\left(\frac{g}{f}\right)$$

(See Fig 1), then the Index of C with respect to the flow is defined as the number of CCW encirlements of the origin by the tangent vector of the flow as the curve is traversed in a CCW direction - as given in Eq (1). Fig 1 shows how this might be down graphically. It can also be done by integration - see Eqn (2).

$$I_C = \frac{\Delta\phi}{2\pi} \tag{1}$$

$$= \frac{1}{2\pi}\oint d\phi = \frac{1}{2\pi}\oint d\left(\arctan\left(\frac{g}{f}\right)\right) = \frac{1}{2\pi}\oint \frac{fdg - gdf}{f^2+g^2} \tag{2}$$

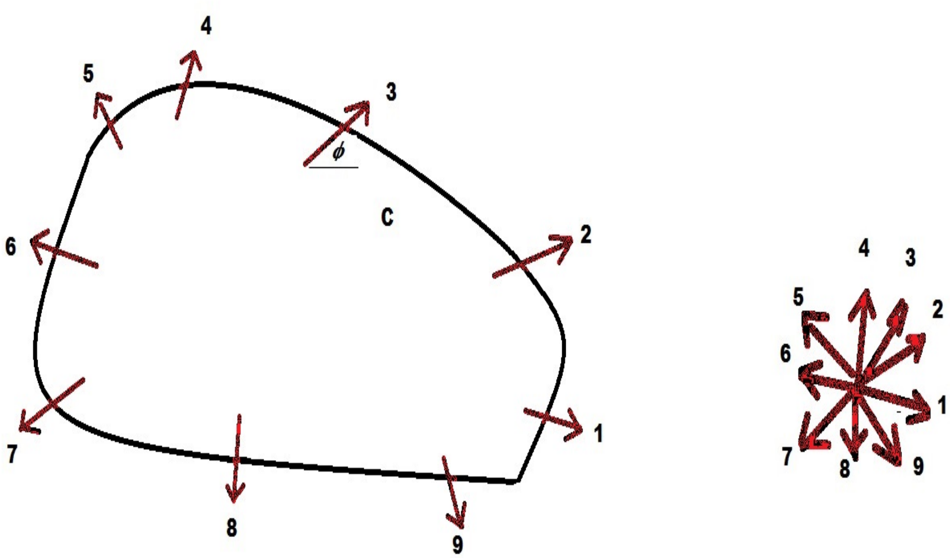

Figure 1: Tangent vectors along C as it is traversed CCW: Here, $I_C = +1$.

Exercise: What happens to the index of the curve in Fig 1 if each tangent vector is reversed in direction?

Properties of Indices

(1) Deform C continuously to C' without going through a fixed point, then $I_C = I_{C'}$.

(2) If C does not encircle a fixed point, then $I_C = 0$.

(3) If t is replaced by $-t$, then I_C does not change (see exercise above).

(4) If C is a PO, then $I_C = +1$ - See Fig 2. So C must encircle at least one fixed point.

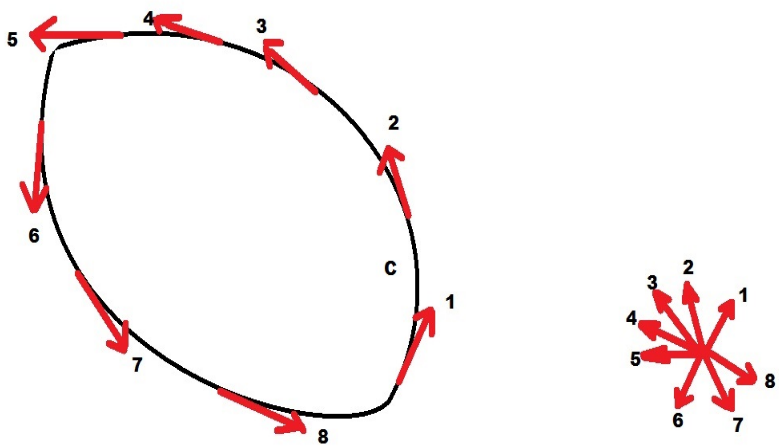

Figure 2: Tangent vectors for a PO: Here, $I_C = +1$.

Index of a point
Deform C continuously till all that is "inside the curve is the point". The index of the point is then the value of I_C. Then it follows that

- Index of a node is +1. See Fig 3
- Index of a spiral is +1. This has a similar diagram to the node.
- Index of a saddle is -1. See Fig 3

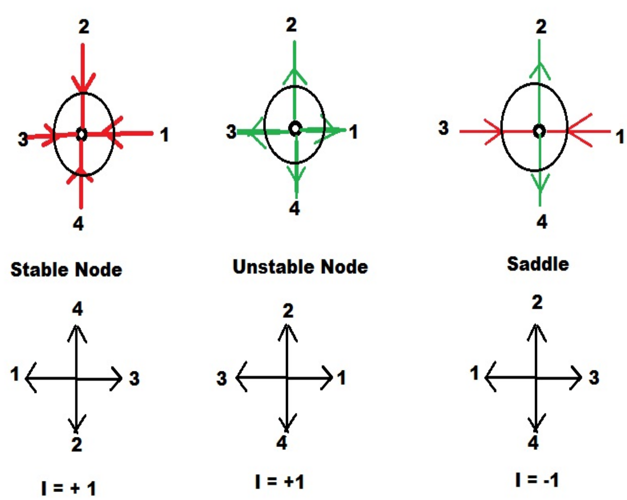

Figure 3: Indices for Node and Saddle

Property (1) and the fact that the index of individual fixed points can be determined enables us to conclude Eqn (3) holds - see Fig 4.

$$I_C = I_{C'} = \sum_p I_p \tag{3}$$

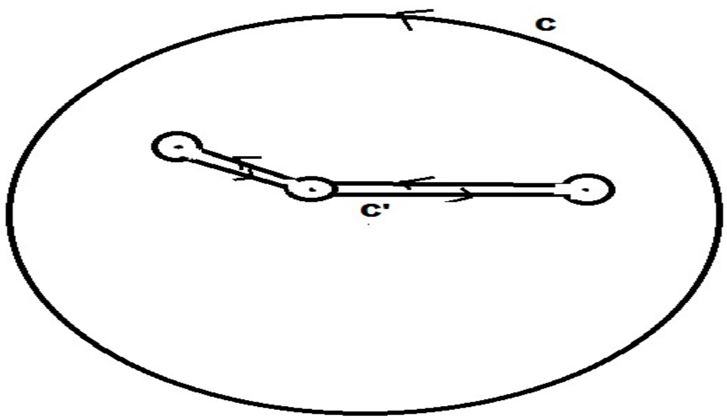

Figure 4: Curve and the fixed points it encircles

Putting this all together gives

Theorem 1. *If C is a PO, then C must encircle fixed points whose indices sum to +1.*

2 Gradient Systems

If the flow is a gradient system, i.e. if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = -\nabla V \stackrel{\triangle}{=} -\frac{\partial V}{\partial \mathbf{x}}, \tag{4}$$

then there can be no POs.
proof: On the contrary, assume that there is a PO (of period T). Then

$$0 = \Delta V = \int_0^T \frac{dV}{dt}\, dt = \int_0^T \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \dot{\mathbf{x}}\, dt = -\int_0^T ||\dot{\mathbf{x}}||^2\, dt < 0$$

i.e. a contradiction. Hence the result follows.

Corollary: No 1-d system has a PO. Proof: Every 1-d system is a gradient system.

3 *Lyapunov* Function

If there exists a global strict *Lyapunov* function for the fixed point $\mathbf{x}_e$ of the flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then by definition all orbits converge to $\mathbf{x}_e$, and hence there can be no POs.

4 *Dulac*'s criterion

Dulac's criterion gives a sufficient condition for ruling out the existence of PO for a 2-d flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

Theorem 2. *If*

(1) R is a simply connected region in $\mathbb{R}^2$ - informally a region with no holes,

(2) $\mathbf{f} = (f, g)^T$ is continuously differentiable in R,

(3) there exists a continuously differentiable function $\beta : R \to \mathbb{R}$

such that

$$\nabla \cdot (\beta \mathbf{f}) \triangleq \frac{\partial}{\partial x}[\beta f] + \frac{\partial}{\partial y}[\beta g] \tag{5}$$

does not change sign in R, then there is no PO in R.

proof: On the contrary, assume that there is a PO C enclosing an area A in the region R (see Fig. 5) for which, wlog,$\nabla \cdot (\beta \mathbf{f}) > 0$. Then, using *Green*'s Theorem,

$$0 < \underbrace{\iint_A \nabla \cdot (\beta \mathbf{f}) \, da = \oint_C (\beta \mathbf{f}) \cdot \mathbf{n} \, dl}_{\text{Green's Theorem}} = \oint_C \beta (\dot{\mathbf{x}} \cdot \mathbf{n}) \, dl = 0,$$

i.e. a contradiction. A similar contradiction occurs if $\nabla \cdot (\beta \mathbf{f}) < 0$ in R. The theorem follows.

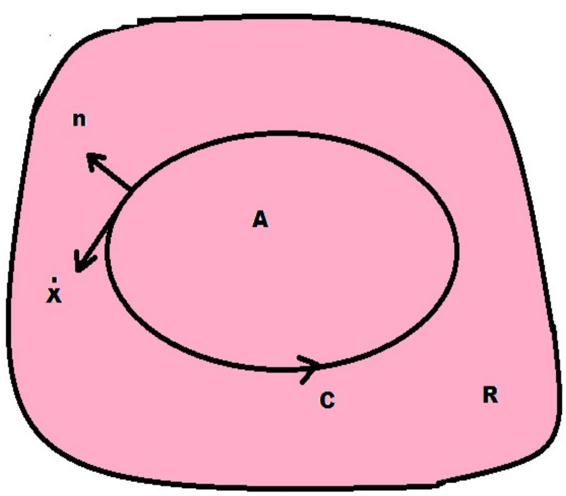

Figure 5: PO C encloses area A within region R; $\dot{\mathbf{x}}$ is tangential to and $\mathbf{n}$ is normal to C.

There is no general method of choosing β. Some of the more commonly used functions are

- $\beta(x, y) = 1$. The criterion was originally derived for this case by *Bendixson*.
- $\beta(x, y) = 1/xy$ or $1/x^a y^b$. Used in population models where $x, y \geq 0$.
- $\beta(x, y) = e^{ax}$ or e^{by}.