Poincaré-Bendixson Theorem

The *Poincaré-Bendixson* (P-B) Theorem gives sufficient conditions for the existence of a periodic orbit (PO) in a 2-d flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. The usual formulation of the theorem is:-

Theorem 1. *If*

(1) R is a closed bounded subset of $\mathbb{R}^2$,

(2) $\mathbf{f}$ is continuously differentiable on an open set that contains R,

(3) R contains no fixed points, and

(4) there exists a trajectory C confined to R,

then either

(i) C is a PO or

(ii) C converges to a PO.

In either case, there is a PO in R.

In practice, the verification of condition (4) is at the heart of determining whether the theorem holds. One way of doing this is to show that R is a *trapping* region, i.e. that the tangent vectors on the boundary of R point into the region.

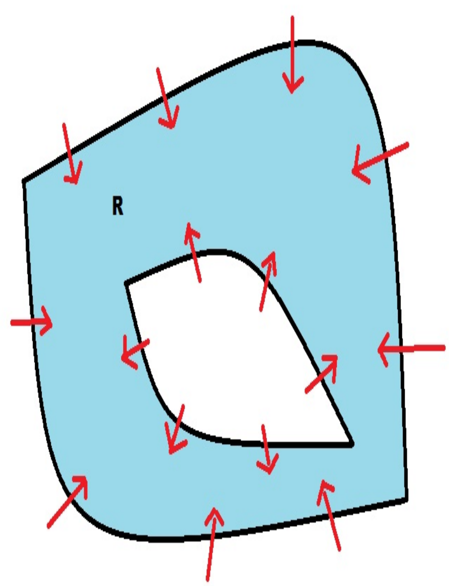

Figure 1: A trapping region R

An alternative formulation of the theorem is:- Let $C = C(t)$ be a trajectory of the flow, and let $\omega(C)$ be the omega limit set of C [1]

Theorem 2. *If C is a bounded trajectory and if $\omega(C)$ contains no fixed points of the flow, then either C is a PO (i.e. $C = \omega(C)$) or $\omega(C)$ consists of a PO to which C converges.*

What other possible behaviours exist for 2-d flows in closed bounded regions? Let's assume that conditions (1), (2) and (4) of Theorem 1 hold, and condition (3) is modified to allow a finite number of fixed points. Then the following generalised version of the theorem holds.

Theorem 3. *If C is a bounded trajectory and if $\omega(C)$ contains a finite number of fixed points of the flow, then one of the following holds:*

(i) $\omega(C)$ consists of a single (fixed) point,

(ii) $\omega(C)$ consists of a PO,

(iii) $\omega(C)$ is a homoclinic cycle.

[1] Recall that $\omega(C)$ consists of the set of all points p such that there exists a sequence of times t_j with $C(t_j) \to p$.