Poincaré-Bendixson Theorem, *Poincaré* Section and *Poincaré* Map

A 2-d flow has a unique fixed point at the origin. Expressed in polar coordinates, the flow equations are

$$\dot{r} = r\left(1 - r^2(1 + \frac{1}{2}\sin\theta)\right) \tag{1}$$

$$\dot{\theta} = 1 + \frac{1}{2}\sin\theta \tag{2}$$

Does it have a periodic orbit and if so, is the orbit stable?

The existence of the periodic orbit may be established by the *Poincaré-Bendixson* (P-B) theorem. Firstly we note that $\dot{\theta} > 0$ and so the position vector of the flow does (ccw) rotations around the origin, an obvious prerequisite for a periodic orbit.

Since the flow is described in polar coordinates, we'll attempt to find a trapping region that is annular in shape, i.e. $R_i \leq r \leq R_o$. If

$$\begin{cases} \dot{r} < 0, & \text{on } R_o; \\ \dot{r} > 0, & \text{on } R_i. \end{cases}$$

then this is sufficient to make the annulus trapping. To this end, we see that when

$$\begin{aligned} \dot{r} &< 0 \\ \Rightarrow 1 - r^2(1 + \frac{1}{2}\sin\theta) &< 0 \\ \Rightarrow r^2 &> \frac{1}{1 + \frac{1}{2}\sin\theta} \end{aligned} \tag{3}$$

Similarly when

$$\begin{aligned} \dot{r} &> 0 \\ \Rightarrow 1 - r^2(1 + \frac{1}{2}\sin\theta) &> 0 \\ \Rightarrow r^2 &< \frac{1}{1 + \frac{1}{2}\sin\theta} \end{aligned} \tag{4}$$

Furthermore since

$$\frac{2}{3} \leq \frac{1}{1 + \frac{1}{2}\sin\theta} \leq 2$$

we get that when $r > \sqrt{2}$, $\dot{r} < 0$ and when $r < \sqrt{2/3}$, $\dot{r} > 0$. Thus, for a suitably small $\epsilon > 0$, a choice for the annular trapping region is

$$\sqrt{\frac{2}{3}} - \epsilon \leq r \leq \sqrt{2} + \epsilon \tag{5}$$

In this bounded region the flow equations are continuously differentiable and there are no fixed points, hence the P-B theorem enables us to conclude that there is a periodic orbit in the region.

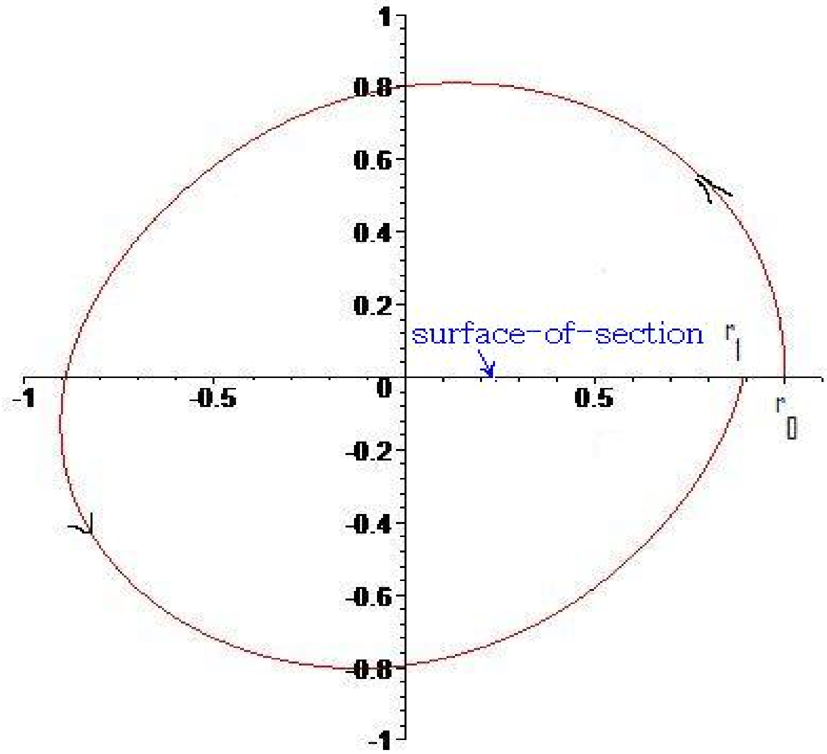

Figure 1: orbit starting at r_0 and returning to r_1.

In order to do some further analysis, take as surface of section the positive x-axis and compute the *Poincaré* map of an orbit starting and ending on this section. [See Fig. 1].

We'll denote the solutions of Equations (1) and (2) with initial conditions $r(0) = r_0,\ \theta(0) = 0$ by $R(t)$ and $\Theta(t)$ respectively. By integrating Equation (2), we see that

$$\begin{aligned} \int_0^t d\tau &= \int_0^{\Theta} \frac{d\theta}{1 + \frac{1}{2}\sin\theta} \\ \Rightarrow \Theta(t) &= 2\arctan\left(\frac{\sqrt{3}}{2}\tan\left(\frac{\sqrt{3}}{4}t + \frac{\pi}{6}\right) - \frac{1}{2}\right) \end{aligned} \tag{6}$$

and for $\Theta = 2\pi$, we get the period of the orbit (T):

$$T = \frac{2\pi}{\sqrt{1 - (\frac{1}{2})^2}} = \frac{4\pi}{\sqrt{3}} \tag{7}$$

We are interested in how $R(t)$ not only varies with t but also with r_0. So representing $R(t)$ as $\mathcal{R}(t, r_0)$, we have that the first-return-to-section map is $r_1 \triangleq R(T) = \mathcal{R}(T, r_0)$ which defines the *Poincaré* map P. Then further iterations are given by

$$r_{k+1} = P(r_k) \triangleq \mathcal{R}(T, r_k) \tag{8}$$

The periodic orbit(s) of the flow described by Equations (1) and (2) are the fixed points of the map of Equation (8) - denoted by r_e. Furthermore, the multiplier of this map is

$$P'(r) = \frac{\partial \mathcal{R}}{\partial r}(T, r) \tag{9}$$

To compute this multiplier, proceed as follows: From Equations(1) and (6), $R(t)$ satisfies

$$\dot{R}(t) = R(t)\left(1 - R^2(t)(1 + \frac{1}{2}\sin\Theta(t))\right), \quad R(0) = r_k$$

Since $R(t) = \mathcal{R}(t, r)$, we can rewrite this equation as

$$\frac{\partial \mathcal{R}}{\partial t}(t, r) = \mathcal{R}(t, r)\left(1 - \mathcal{R}^2(t, r)(1 + \frac{1}{2}\sin\Theta(t))\right), \quad \mathcal{R}(0, r_k) = r_k$$

(notice how the ordinary derivative has been replaced with the appropriate partial derivative). Differentiating this expression with respect to r, and then interchanging the order of the two derivatives, yields

$$\begin{aligned}
\frac{\partial^2 \mathcal{R}}{\partial r \partial t}(t, r) &= \frac{\partial \mathcal{R}}{\partial r}(t, r)\left(1 - \mathcal{R}^2(t, r)(1 + \frac{1}{2}\sin\Theta(t))\right) \\
&\quad + \mathcal{R}(t, r)\left(-2\mathcal{R}(t, r)\frac{\partial \mathcal{R}}{\partial r}(t, r)(1 + \frac{1}{2}\sin\Theta(t))\right), \quad \frac{\partial \mathcal{R}}{\partial r}(0, r_k) = 1 \\
\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{R}}{\partial r}(t, r)\right) &= \left(1 - 3\mathcal{R}^2(t, r)(1 + \frac{1}{2}\sin\Theta(t))\right)\frac{\partial \mathcal{R}}{\partial r}(t, r), \qquad \frac{\partial \mathcal{R}}{\partial r}(0, r_k) = 1
\end{aligned} \tag{10}$$

Since r does not explicitly appear in Equation (10), we can cloak its existence by defining the "sensitivity" $S(t)$ of R with respect to r as $S(t) \triangleq \frac{\partial \mathcal{R}}{\partial r}(t, r)$ and rewriting Equation (10) as the ordinary differential equation

$$\dot{S}(t) = \left(1 - 3R^2(t)(1 + \frac{1}{2}\sin\Theta(t))\right)S(t), \quad S(0) = 1 \tag{11}$$

Thus integrating Equation (11) in conjugation with Equations (1) and (2) over $0 \leq t \leq T$ gives the multiplier of the *Poincaré* map $P'(r) = S(T)$ which can then be evaluated at the fixed point $r = r_e$.

In the accompanying *Maple* worksheet "*Poincaré*-map and Fixed Points", these calculations are carried using a numerical solver to integrate the differential equations.