Computing Invariant Manifolds of 2-d Saddles

Generically in 2-d, the stable and unstable manifolds of a saddle are curves. In a neighbourhood of the fixed point they may be represented by power series. The following discussion concentrates on calculating (recursively) coefficients of these power series. I shall assume that the fixed point is at the origin of the phase plane (if not, change variables to place it at the origin). I shall give two versions of the approach: the first requires less computation but potentially gives a power series with a smaller domain of convergence; the second usually requires a change of variable that aligns the stable and unstable manifolds with the linearised stable and unstable eigendirections respectively. The upside of this extra work is a potentially larger domain of convergence for the power series. Having computed the power series, changing back to the original variables results in an implicit representation of the manifolds.

I'll illustrate the approaches on a specific example:

The 2-d flow

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= 4x + xy \end{aligned} \tag{1}$$

has a fixed point $\mathbf{x}_e = (0,0)^T$ which is saddle since its *Jacobian* matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

(for which the stable eigenvalue is $\lambda^s = -2$ and the unstable eigenvalue $\lambda^u = 2$.)
The task is to find the unstable manifold $W^u(\mathbf{0})$.

Approach 1 Assuming that we can write the unstable manifold (at least locally) as

$$y = h(x) = h^{(0)} + h^{(1)}x + \frac{h^{(2)}}{2!}x^2 + \frac{h^{(3)}}{3!}x^3 + \cdots = \sum_{j=0}^{\infty} \frac{h^{(j)}(0)}{j!}x^j \tag{2}$$

where $h^{(j)}(0)$ is the j-th derivative of h at $x = 0$, we know that

(i) $h^{(0)} = h(0) = 0$ since $(0,0)$ is the fixed point

(ii) $h^{(1)} = Dh(0) = \text{ dir } (\mathbf{e}_{\lambda=2}) = 2$ since we are computing the unstable manifold.

We obtain on differentiating Eq (2) with respect to "t", and substituting from Eq (1)

$$\begin{aligned} \dot{y} &= Dh(x)\dot{x} \\ \Rightarrow \quad 4x + xy &= Dh(x)y \\ \Rightarrow \quad 4x + xh(x) &= Dh(x)h(x) \end{aligned} \tag{3}$$

Evaluating Eq (3) at $x = 0$ yields $0 = 0$. No new information.
Differentiating Eq (3) with respect to "x" gives

$$4 + h(x) + xDh(x) = D^2h(x)h(x) + (Dh(x))^2 \tag{4}$$

Evaluating Eq (4) at $x = 0$ yields $4 = (Dh(0))^2$. Still no new information (alternatively, we would have to choose one or other of the square roots of 4 to continue with the computation - but we know that we want to use $Dh(0) = 2$).
Differentiating Eq (4) with respect to "x" gives

$$2Dh(x) + xD^2h(x) = D^3h(x)h(x) + 3D^2h(x)Dh(x) \tag{5}$$

Evaluating Eq (5) at $x = 0$ yields $4 = 6D^2h(0) \Rightarrow D^2h(0) = 2/3$.
Differentiating Eq (5) with respect to "x" gives

$$3D^2h(x) + xD^3h(x) = D^4h(x)h(x) + 4D^3h(x)Dh(x) + 3\left(D^2h(x)\right)^2 \tag{6}$$

Evaluating Eq (6) at $x = 0$ yields $2 = 8D^3h(0) + 4/3 \Rightarrow D^3h(0) = 1/12$.

We can continue in this fashion evaluating as many coefficients as are deemed necessary.

We have determined that $W^u(\mathbf{0})$ is given locally by

$$y = 2x + \frac{1}{3}x^2 + \frac{1}{72}x^3 + O(x^4) \tag{7}$$

Approach 2 A system with $x_e = \mathbf{0}$ can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = A\mathbf{x} + g(\mathbf{x}) \tag{8}$$

where A is the *Jacobian* matrix of $\mathbf{f}$ evaluated at $x_e = \mathbf{0}$, and $g(.)$ contains the higher order terms. If the fixed point is a 2-d saddle with eigenvalues λ^s and λ^u, then denoting the corresponding eigenvectors by e^s and e^u respectively, we can define the invertible matrix

$E = [e^s,\ e^u]$. Then the transformation or change of variables $\mathbf{x} = E\mathbf{X}$ results in the new representation of the system:

$$\dot{\mathbf{x}} = A\mathbf{x} + g(\mathbf{x}) \qquad \overset{\mathbf{x}=E\mathbf{X}}{\rightarrow} \qquad \dot{\mathbf{X}} = \Lambda\mathbf{X} + G(\mathbf{X}) \tag{9}$$

where $\Lambda = E^{-1}AE = diag\{\lambda^s,\ \lambda^u\}$ and $G = E^{-1} \circ g \circ E$.

The specific example of Eq (1) can be rewritten as

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ xy \end{pmatrix}$$

Now, we compute

$$\lambda^s = -2, \qquad \lambda^u = 2, \qquad e^s = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad e^u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}, \qquad E^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

thus the transformed system is

$$\dot{\mathbf{X}} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{X} + \frac{X^2 - Y^2}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or

$$\dot{X} = -2X + \frac{X^2 - Y^2}{2} \tag{10}$$

$$\dot{Y} = 2Y - \frac{X^2 - Y^2}{2} \tag{11}$$

With this choice of new variables, we know that W^s is tangential to X-axis at the origin, while W^u is tangential to Y-axis at the origin. Thus to describe W^u by a power series in the neighbourhood of the origin (see Fig. 1), we should write

$$X = H(Y) = H^{(0)} + H^{(1)}Y + \frac{H^{(2)}}{2!}Y^2 + \frac{H^{(3)}}{3!}Y^3 + \cdots = \sum_{j=0}^{\infty} \frac{H^{(j)}(0)}{j!} Y^j \tag{12}$$

where $H^{(j)}(0)$ is the j-th derivative of H at $Y = 0$; we know that

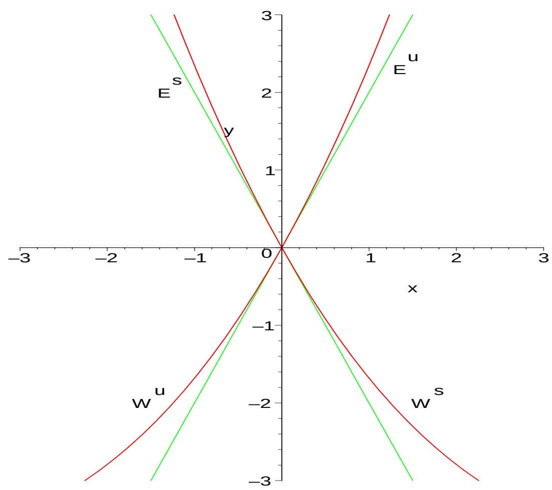

Figure 1: Invariant manifolds & linearised invariant manifolds for a 2-d Saddle

(I) $H^{(0)} = H(0) = 0$ since $(0,0)$ is the fixed point

(II) $H^{(1)} = DH(0) = 0$ since the unstable manifold is tangential to Y-axis.

To compute more coefficients of H, we proceed in similar fashion to Approach 1. Differentiating the leftmost equality of Eq (12) with respect to t and substituting from Eqs (10) and (11) gives

$$\begin{aligned} \dot{X} &= DH(Y)\dot{Y} \\ \Rightarrow \qquad -2X + \frac{X^2 - Y^2}{2} &= DH(Y)\left[2Y - \frac{X^2 - Y^2}{2}\right] \\ \Rightarrow \ -2H(Y) + \frac{H^2(Y) - Y^2}{2} &= DH(Y)\left[2Y - \frac{H^2(Y) - Y^2}{2}\right] \end{aligned} \tag{13}$$

Evaluating Eq (13) at $Y = 0$ yields no new information. Repeatedly differentiating

Eq (13) with respect to Y and evaluating the resultant expressions at $Y = 0$ recursively generates the coefficients of the power series. My calculations yield

$$X = -\frac{1}{12}Y^2 + \frac{1}{96}Y^3 + O(Y^4) \tag{14}$$

or in terms of the original variables

$$\frac{2x - y}{4} = -\frac{1}{12}\left(\frac{2x + y}{4}\right)^2 + \frac{1}{96}\left(\frac{2x + y}{4}\right)^3 + O(\left(\frac{2x + y}{4}\right)^4) \tag{15}$$