Examples of Trapping Region calculation for use with *Poincaré-Bendixson* Theorem

Consider the 2-d flow, which has a unique fixed point at the origin,

$$\begin{aligned}\dot{x} &= -x(x^2+y^2-2\alpha x-\beta)-y \\ \dot{y} &= -y(x^2+y^2-2\alpha x-\beta)+x\end{aligned}$$

1 Converting to Polar Coordinates

This gives

$$\begin{aligned}\dot{r} &= -r(r^2-2\alpha r\cos\theta-\beta) \qquad (1) \\ \dot{\theta} &= 1\end{aligned}$$

The existence of a periodic orbit may be established by the *Poincaré-Bendixson* (P-B) theorem. Firstly we note that $\dot{\theta} > 0$ and so the position vector of the flow does (ccw) rotations around the origin, an obvious prerequisite for a periodic orbit.
Since the flow is described in polar coordinates, we'll attempt to find a trapping region that is annular in shape, i.e. $R_{\text{inner}} \leq r \leq R_{\text{outer}}$. If

$$\begin{cases} \dot{r} > 0, & \text{on } R_{\text{inner}}, \\ \dot{r} < 0, & \text{on } R_{\text{outer}}, \end{cases}$$

then this is sufficient to make the annulus trapping.

1.1 Determining R_{inner}

From Eq (1), we get that

$$\dot{r} > 0 \Leftrightarrow r^2 - 2\alpha r\cos\theta - \beta < 0$$

This latter inequality can only be achieved for all θ if $\beta \geq 0$ which we'll assume from here on. The inequality can be solved to get

$$\alpha\cos\theta - \sqrt{\alpha^2\cos^2\theta+\beta} < r < \alpha\cos\theta + \sqrt{\alpha^2\cos^2\theta+\beta}$$

This can be achieved if r is chosen so that $r < R_m$ where

$$R_m = \min_{\theta}\{\alpha\cos\theta + \sqrt{\alpha^2\cos^2\theta+\beta}\} = -|\alpha| + \sqrt{\alpha^2+\beta}. \qquad (2)$$

(Exercise: show that $\alpha\cos\theta - \sqrt{\alpha^2\cos^2\theta+\beta} < R_m$.)

1.2 Determining R_{outer}

Again, from Eq (1), we get that

$$\dot{r} < 0 \Leftrightarrow r^2 - 2\alpha r \cos\theta - \beta > 0$$

This inequality above can be solved to get

$$r < \alpha\cos\theta - \sqrt{\alpha^2\cos^2\theta + \beta} \qquad \text{or} \qquad \alpha\cos\theta + \sqrt{\alpha^2\cos^2\theta + \beta} < r$$

And, since $R_{\text{outer}} > R_{\text{inner}}$, we can achieve this by choosing $r > R_M$ where

$$R_M = \max_{\theta}\{\alpha\cos\theta + \sqrt{\alpha^2\cos^2\theta + \beta}\} = |\alpha| + \sqrt{\alpha^2 + \beta}. \tag{3}$$

Thus, for any suitably small $\epsilon > 0$, a choice for the annular trapping region is

$$\underbrace{R_m - \epsilon}_{R_{\text{inner}}} \leq r \leq \underbrace{R_M + \epsilon}_{R_{\text{outer}}} \tag{4}$$

In this bounded region the flow equations are continuously differentiable and there are no fixed points, hence the P-B theorem enables us to conclude that there is a periodic orbit in the region.

2 Energy approach

Taking as "energy" function

$$V = \frac{1}{2}x^2 + \frac{1}{2}y^2 \tag{5}$$

$$\Rightarrow \dot{V} = (x^2 + y^2)q(x, y) \tag{6}$$

where

$$q(x, y) = (\alpha^2 + \beta) - (x - \alpha)^2 - y^2 \tag{7}$$

Note that $q = 0$ describes a circle, and that $\dot{V} > 0$ is the region inside the circle and $\dot{V} < 0$ the region outside the circle. Moreover, defining $V_{\text{inner}} \triangleq 2(\sqrt{\alpha^2 + \beta} - |\alpha|)$ and $V_{\text{outer}} \triangleq 2(\sqrt{\alpha^2 + \beta} + |\alpha|)$, the level sets $V = V_{\text{inner}} - \epsilon$ and $V = V_{\text{outer}} + \epsilon$ (see Fig 1) are respectively inside and outside the circle, and therefore have increasing and decreasing energy respectively along trajectories of the system. Hence the region between these two level sets (the yellow annulus in Fig 1) must be a trapping region. As before since this annulus does not contain any fixed points, the P-B theorem tells us that there must be a periodic orbit contained in the region.

Note: Due to the choice of $V = (1/2)r^2$ and the fact that $q = 0$ turned out to be a circle, the annular regions from both methods happened to be the same.

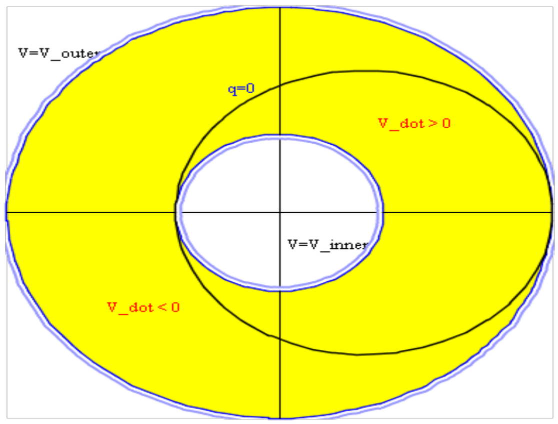

Figure 1: Trapping Region in yellow

3 Polygonal Trapping Regions

The *Sel'kov* model of glycolysis is

$$\dot{x} = -x + ay + x^2y \tag{8}$$
$$\dot{y} = b - ay - x^2y \tag{9}$$

where $x(t)$ and $y(t)$ are the concentrations of ADP (adenosine diphosphate) and F6P (fructose-6 phosphate) and $a, b > 0$ are parameters.

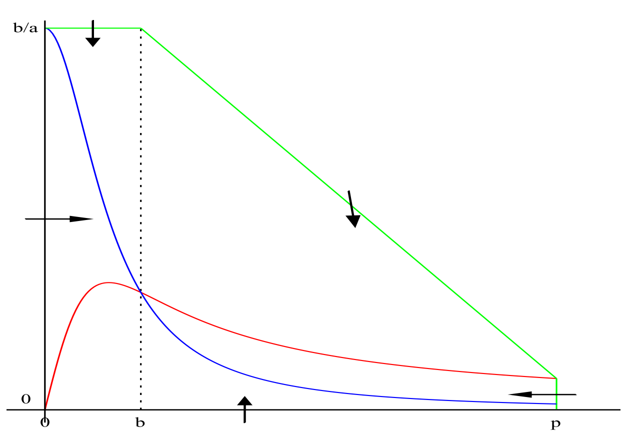

Figure 2: Nullclines and Tangent Vectors.(red = x-nullcline, blue = y-nullcline)

Consider the boundary of the irregular pentagon:

1. On $x = 0,\ \dot{x} = ay > 0$ whenever $y > 0$
2. On $x = p,\ \dot{x} = -p + (p^2 + a)y < 0$ whenever $y < p/(p^2 + a)$. This is true since y lies below the red nullcline.
3. On $y = 0,\ \dot{y} = b > 0$
4. on $y = b/a,\ \dot{y} = -x^2(b/a) < 0$
5. The sloped line is drawn with slope -1, i.e. along the line, $\dot{y} + \dot{x} = 0$ [1]. On the other hand, the tangent vector of the flow satisfies $\dot{y} + \dot{x} = b - x < 0$ whenever $x > b$; hence the tangent vector is more "upright" than a vector lying along the line, i.e. its orientation is as indicated in Fig 2.

Therefore the flow directions are as shown and the region is trapping.
The pentagonal region contains the system's only fixed point - at the intersection of the nullclines. In order to apply the P-B theorem, what needs to be done?

[1] This is motivated by noting that for large x, $\dot{y}/\dot{x} \approx -1$