FOURIER SERIES

1. Periodic Functions

Recall that f has period T if $f(x+T) = f(x)$ for all x. If f and g are periodic with period T, then if $h(x) = af(x) + bg(x)$, where a and b are constants,

$$\begin{aligned} h(x+T) &= af(x+T) + bg(x+T) \\ &= af(x) + bg(x) = h(x) \end{aligned}$$

and thus h also has period T.

1.1. Functions of period 2π. We want to represent general functions of period 2π in terms of cosine and sine functions which also have period 2π, i.e., by the *trigonometric series*:

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

where a_0, a_1, a_2, $\dots$ and b_1, b_2, b_3, $\dots$ are real constants, the *coefficients* of the series.

Clearly, since each term of this infinite series has period 2π, *if it converges* its sum will also have period 2π.

Assume then that we can write $f(x)$ as

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We have to determine the coefficients:

- To determine a_0:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\, dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\, dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\, dx \\ &= \pi a_0 + 0 + 0 \end{aligned}$$

  Thus, $a_0 = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\, dx$

- To determine a_n:

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos mx\cos nx\,dx$$
$$+\sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\cos mx\sin nx\,dx$$

REMARK 1.1.

$$\begin{aligned}\int_{-\pi}^{\pi}\cos mx\cos nx\,dx &= \frac{1}{2}\left[\int_{-\pi}^{\pi}\cos(m+n)x\,dx + \int_{-\pi}^{\pi}\cos(m-n)\,dx\right]\\ &= \frac{1}{2}\left\{\left[\frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} + \left[\frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}\right\}\\ &= 0\end{aligned}$$

unless $n = m$, in which case

$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = \pi$$

Similarly

$$\begin{aligned}\int_{-\pi}^{\pi}\cos mx\sin nx\,dx &= \frac{1}{2}\left[\int_{-\pi}^{\pi}\sin(m+n)x\,dx + \int_{-\pi}^{\pi}\sin(n-m)\,dx\right]\\ &= -\frac{1}{2}\left\{\left[\frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} + \left[\frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi}\right\}\\ &= 0\end{aligned}$$

(even when $m = n$ Thus

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \pi a_m$$

and so

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

and similarly

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

These formulae for the coefficients a_0, a_n and b_n are called the *Euler formulae.* Note that the formula for a_0 is formally the same as that for a_n and so we can write:

$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\end{aligned}}$$

Note If $f(x)$ is periodic with period 2π then $\int_{-\pi}^{\pi} f(x)\,dx$ can be integrated over any interval of length 2π, e.g., $\int_0^{2\pi} f(x)\,dx$

The expression

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called the *Fourier series* for $f(x)$ with *Fourier coefficients* a_0, a_n and b_n.

Example. Square wave

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases} \qquad f(x + 2\pi) = f(x)$$

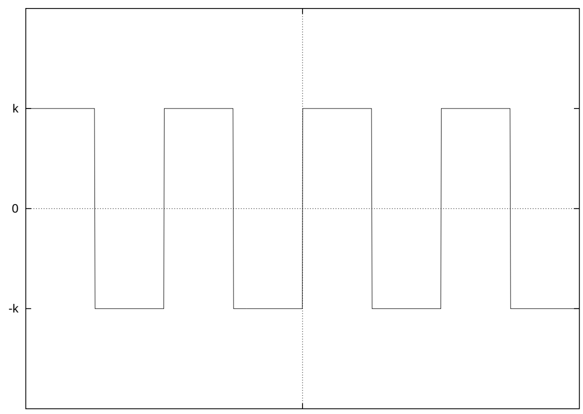


FIGURE 1. $f(x)$

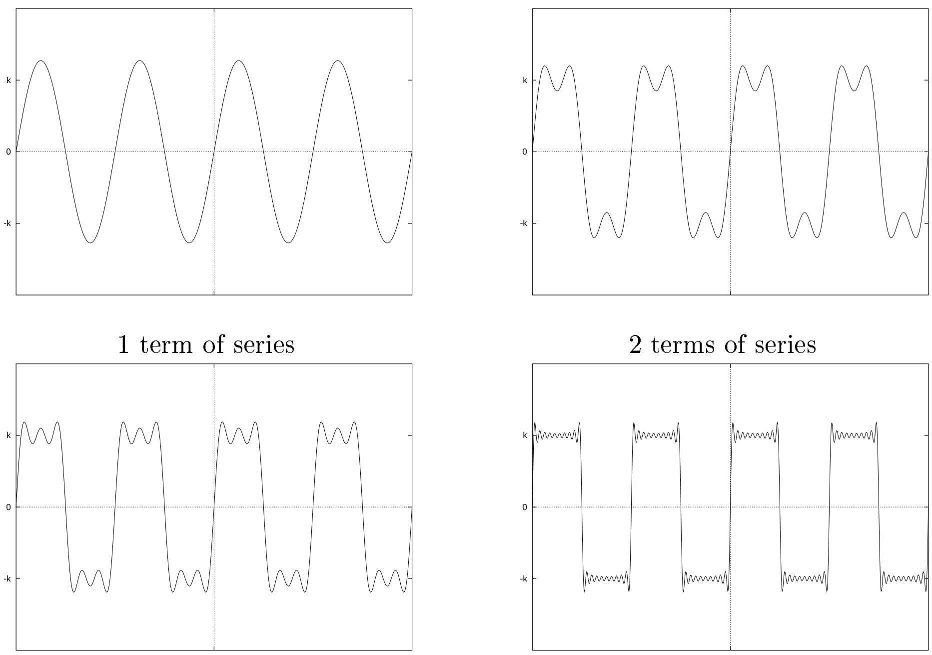
$$a_0 = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_0^{\pi} k\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left\{\left[-\frac{k}{n}\sin nx\right]_{-\pi}^{0} + \left[\frac{k}{n}\sin nx\right]_0^{\pi}\right\} \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right] \\
&= \frac{1}{\pi}\left\{\left[\frac{k}{n}\cos nx\right]_{-\pi}^{0} - \left[\frac{k}{n}\cos nx\right]_{0}^{\pi}\right\} \\
&= \frac{k}{n\pi}\left\{\cos 0 - \cos(-n\pi) - \cos(n\pi) + \cos 0\right\} \\
&= \frac{2k}{n\pi}\left\{1 - \cos n\pi\right\} \\
&= \frac{2k}{n\pi}\left[1 - (-1)^n\right] \\
&= \begin{cases} 0, & n\text{even} \\ \frac{4k}{n\pi}, & n\text{odd} \end{cases}
\end{aligned}
$$

Thus, $b_1 = \frac{4k}{\pi}$, $b_2 = 0$, $b_3 = \frac{4k}{3\pi}$, $b_4 = 0$, etc. and the Fourier series is

$$
\frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right]
$$



1 term of series

2 terms of series

3 terms of series

10 terms of series

FIGURE 2. Fourier series for $f(x)$

Formal definition. If $f(x)$ is periodic, with period 2π, piecewise continuous in $[-\pi, \pi]$, has a left and right-hand derivative at each $x_0 \in [-\pi, \pi]$, then its Fourier series is convergent to $f(x)$, except at a discontinuity x_0, where the sum is $\frac{1}{2}[f(x_0+0) + f(x_0 - 0)]$.

REMARK 1.2. The left hand derivative of f at x_0 is

$$\lim_{h \to 0_-} \frac{f(x_0 - 0) - f(x_0 - h)}{h}$$

where $f(x_0 - 0) = \lim_{x \to x_{0_-}} f(x)$ and similarly the right hand derivative of f at x_0 is

$$\lim_{h \to 0_+} \frac{f(x_0 + h) - f(x_0 + 0)}{h}$$

We can in these circumstances write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Example. Saw-tooth wave.

$$f(x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$$

with period 2π

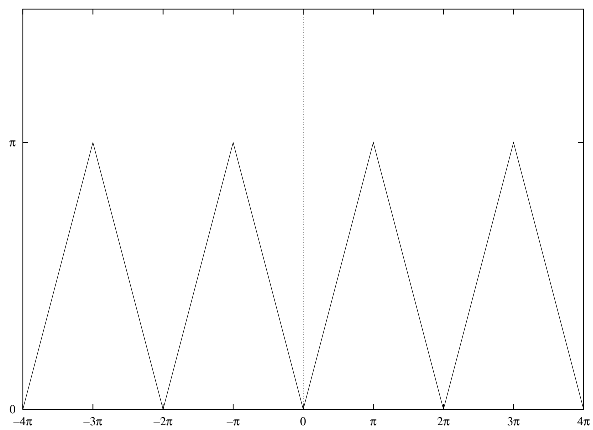


FIGURE 3. $f(x)$

$$\begin{aligned} a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx &= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-x)\, dx + \int_{0}^{\pi} x\, dx \right] \\ &= \frac{2}{\pi} \int_{0}^{\pi} x\, dx \\ &= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \pi \end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right] \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx \\
&= \frac{2}{\pi}\left\{\left[\frac{x}{n}\sin nx\right]_0^{\pi} - \frac{1}{n}\int_{0}^{\pi}\sin nx\,dx\right\} \\
&= \frac{2}{\pi}\left\{0 + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right\} \\
&= \frac{2}{n^2\pi}\left[\cos n\pi - 1\right] \\
&= \begin{cases} 0, & n \text{even} \\ -\frac{4}{n^2\pi}, & n \text{odd} \end{cases}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right] \\
&= 0
\end{aligned}$$

Thus

$$\begin{aligned}
f(x) &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{\cos nx}{n^2} \\
&= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \dots\right]
\end{aligned}$$

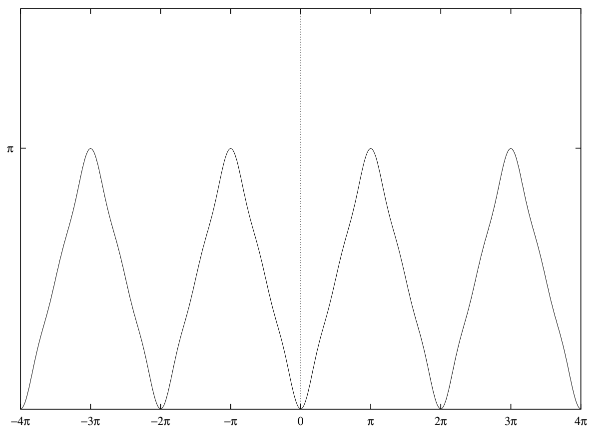


FIGURE 4. First 4 terms of Fourier series for $f(x)$

REMARK 1.3. If $f(x)$ and $g(x)$ each have Fourier series expansions, then the Fourier series for the sum $f(x) + g(x)$ is just the sum of the two expansions.

For example, the Fourier series of the square wave $f(x) = \begin{cases} k, & 0 < x < \pi \\ -k, & -\pi < x < 0 \end{cases}$ with period 2π was shown to be

$$\frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right]$$

Thus the Fourier series for $g(x) = \begin{cases} 2k, & 0 < x < \pi \\ 0, & -\pi < x < 0 \end{cases}$ with period 2π is

$$k + \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right]$$

1.2. Functions of arbitrary period. Let $f(x)$ have period 2ℓ. If we make the substitution $u = \frac{\pi}{\ell}x$, then, when$x = \pm\ell$, $u = \pm\pi$ and so f is periodic in the new variable u with period 2π and thus if f has a Fourier series, then

$$f(x) = f\left(\frac{\ell}{\pi}u\right) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nu + b_n \sin nu)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{\ell}{\pi}u\right)\cos nu\, du$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{\ell}{\pi}u\right)\sin nu\, du$$

or, in terms of x, where $u = \frac{\pi}{\ell}x \Rightarrow du = \frac{\pi}{\ell}\,dx$

$$\boxed{\begin{aligned} a_n &= \tfrac{1}{\ell}\textstyle\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx \\ b_n &= \tfrac{1}{\ell}\textstyle\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx \end{aligned}}$$

and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

Example. Rectified sine wave.

$$f(x) = \sin \pi x \quad 0 < x < 1$$
$$f(x+1) = f(x)$$

Note: $\ell = \frac{1}{2}$.

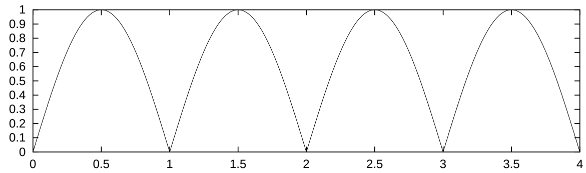


FIGURE 5. $f(x) = |sin\pi x|$

$$\begin{aligned} a_0 &= 2\int_{-1/2}^{1/2} f(x)\,dx = 2\int_0^1 f(x)\,dx \\ &= 2\int_0^1 \sin\pi x\,dx = -\frac{2}{\pi}\left[\cos\pi x\right]_0^1 \\ &= \frac{4}{\pi} \end{aligned}$$

$$\begin{aligned} a_n &= 2\int_0^1 \sin\pi x\cos 2n\pi x\,dx \\ &= \int_0^1 \sin(2n+1)\pi x\,dx + \int_0^1 \sin(2n-1)\pi x\,dx \\ &= -\frac{1}{(2n+1)\pi}\left[\cos(2n+1)\pi x\right]_0^1 - \frac{1}{(2n-1)\pi}\left[\cos(2n-1)\pi x\right]_0^1 \\ &= -\frac{1}{(2n+1)\pi}(-2) - \frac{1}{(2n-1)\pi}(-2) \\ &= \frac{2}{(2n+1)\pi} + \frac{2}{(2n-1)\pi} = \frac{8n}{(4n^2-1)\pi} \end{aligned}$$

$$\begin{aligned} b_n &= 2\int_0^1 \sin\pi x\sin 2n\pi x\,dx \\ &= \int_0^1 \cos(2n-1)\pi x\,dx - \int_0^1 \cos(2n+1)\pi x\,dx \\ &= \frac{1}{(2n-1)\pi}\left[\sin(2n-1)\pi x\right]_0^1 - \frac{1}{(2n+1)\pi}\left[\sin(2n+1)\pi x\right]_0^1 \\ &= 0 \end{aligned}$$

So,

$$\begin{aligned} f(x) &= \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{8n}{(4n^2-1)\pi}\cos 2n\pi x \\ &= \frac{2}{\pi} + \frac{8}{\pi}\left[\frac{\cos 2\pi x}{3} + \frac{2\cos 4\pi x}{15} + \dots\right] \end{aligned}$$

REMARK 1.4. In each of the last two examples $f(x) = f(-x)$ for all x and in each case the coefficients b_n turned out to be zero. Was this a coincidence or can some work be saved by making use of this property?

1.3. Even and odd functions. The function f is said to be *even* if $f(-x) = f(x)$ for all x. For example, the functions $\cos x$, 1, x^2, x^4 are even,
Similarly, f is said to be *odd* if $f(-x) = -f(x)$ for all x. For example, the functions $\sin x$, x, x^3 are odd.If $f(x)$ is even then

$$\begin{aligned}\int_{-k}^{k} f(x)\,dx &= \int_{-k}^{0} f(x)\,dx + \int_{0}^{k} f(x)\,dx \\ (u=-x,\ f(x)=f(-u)=f(u)) &= \int_{0}^{k} f(u)\,du + \int_{0}^{k} f(x)\,dx \\ (du=-dx) &= 2\int_{0}^{k} f(x)\,dx\end{aligned}$$

Similarly, if $f(x)$ is odd, them $\int_{-k}^{k} f(x)\,dx = 0$ Note that if $f(x)$ is <u>even</u> and $g(x)$

is <u>odd</u> then if $h(x) = f(x)g(x)$, $h(-x) = f(x)[-g(x)] = -h(x)$ and so h is <u>odd</u>.
Thus, if $f(x)$ is even, the product $f(x)\sin\frac{n\pi x}{\ell}$ is odd and hence $b_n = 0$. Similarly, if $f(x)$ is odd, the product $f(x)\cos\frac{n\pi x}{\ell}$ is odd and hence $a_n = 0$, and obviously $a_0 = 0$ also. Thus, we have the following:
The Fourier series of an even function $f(x)$ of period 2ℓ is a *Fourier cosine series*:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{\ell}x$$

where

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

The Fourier series of an odd function $f(x)$ of period 2ℓ is a *Fourier sine series*:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{\ell}x$$

where

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

2. Half-range expansions.

Even though Fourier series were introduced for periodic functions on $(-\infty, \infty)$, we can also find Fourier series for functions defined on an interval $[0, \ell]$, by finding either even or odd periodic extensions of period 2ℓ as follows:

2.1. Periodic extensions.

DEFINITION 2.1. The *even periodic extension* $f_1(x)$ of period 2ℓ of the function $f(x)$ defined on $[0, \ell]$ is

$$f_1(x) = \begin{cases} f(x), & 0 \leq x \leq \ell \\ f(-x), & -\ell \leq x \leq 0 \end{cases}$$

with $f_1(x + 2\ell) = f(x)$

$f_1(x)$ has a Fourier cosine series:

$$f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{\ell} x$$

where

$$a_n = \frac{2}{\ell} \int_0^\ell f(x) \cos \frac{n\pi x}{\ell}\, dx$$

This can now be regarded as a Fourier cosine series for $f(x)$ on the interval $0 \leq x \leq \ell$ if we restrict x to that interval, i.e.,

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

for $0 \leq x \leq \ell$

DEFINITION 2.2. The *odd periodic extension* $f_2(x)$ of period 2ℓ of the function $f(x)$ defined on $[0, \ell]$ is

$$f_2(x) = \begin{cases} f(x), & 0 < x < \ell \\ -f(-x), & -\ell < x < 0 \end{cases}$$

with $f_2(x + 2\ell) = f(x)$

$f_2(x)$ has a Fourier sine series:

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{\ell} x$$

where

$$b_n = \frac{2}{\ell} \int_0^\ell f(x) \sin \frac{n\pi x}{\ell}\, dx$$

This can now be regarded as a Fourier sine series for $f(x)$ on the interval $0 \leq x \leq \ell$ if we restrict x to that interval, i.e.,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

for $0 < x < \ell$

Example.

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \end{cases}$$

Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \quad 0 \le x \le 2$$

where

$$\begin{aligned} a_0 &= \int_0^2 f(x)\,dx \\ &= \left\{ \int_0^1 x\,dx + \int_1^2 (2-x)\,dx \right\} \\ &= \left\{ \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 \right\} \\ &= \left\{ \frac{1}{2} + (4-2) - (2 - \frac{1}{2}) \right\} \\ &= \left\{ \frac{1}{2} + \frac{1}{2} \right\} = 1 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx \\ &= \int_0^1 x\cos\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\cos\frac{n\pi x}{2}\,dx \end{aligned}$$

$$\begin{array}{cccccc} u = x & dv = \cos\frac{n\pi x}{2}\,dx & | & u = (2-x) & dv = \cos\frac{n\pi x}{2}\,dx \\ du = dx & v = \frac{2}{n\pi}\sin\frac{n\pi x}{2} & | & du = -dx & v = \frac{2}{n\pi}\sin\frac{n\pi x}{2} \end{array}$$

$$\begin{aligned} \Rightarrow a_n &= \frac{2}{n\pi}\left[x\sin\frac{n\pi x}{2}\right]_0^1 - \frac{2}{n\pi}\int_0^1 \sin\frac{n\pi x}{2}\,dx \\ &+ \frac{2}{n\pi}\left[(2-x)\sin\frac{n\pi x}{2}\right]_1^2 + \frac{2}{n\pi}\int_1^2 \sin\frac{n\pi x}{2}\,dx \\ &= \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_0^1 \\ &- \frac{2}{n\pi}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_1^2 \\ &= \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1 - \cos n\pi + \cos\frac{n\pi}{2}\right) \\ &= \frac{4}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right) \end{aligned}$$

Note that

$$\cos\frac{n\pi}{2} = \begin{cases} 0, & n \text{ odd} \\ 1, & n/2 \text{ even} \\ -1, & n/2 \text{ odd} \end{cases}$$

Thus, n odd $\Rightarrow a_n = 0$ n even and $n/2$ even $\Rightarrow a_n = \dfrac{4}{n^2\pi^2}(2-2) = 0$ n even and

$n/2$ odd $\Rightarrow a_n = \dfrac{4}{n^2\pi^2}(-2-2) = -\dfrac{16}{n^2\pi^2}$

3. Fourier series solutions of differential equations

There is an immediate application of Fourier series in the solution of constant coefficient second order linear differential equations, i.e., equations of the form

$$y'' + ay' + by = R(x)$$

where a and b are constants. The solution of this equation can be written as

$$y(x) = y_h(x) + y_p(x)$$

the sum of the solution of the homogeneous equation $y_h'' + ay_h' + by_h = 0$ and a particular integral or solution $y_p(x)$ of the original differential equation. Recall that if $R(x)$ is a constant or a cosine or sine term, the particular solution of this equation can be found by the method of undetermined coefficients. Thus, if $R(x)$ can be expanded in a Fourier series, the solution can also be found in same way.

Example

$$y'' + y' = r(x)$$

where

$$r(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases}$$

To find y_h the solution of the homogeneous equation

$$y_h'' + y_h' = 0$$

let $y_h = e^{\lambda x}$; substituting we obtain $(\lambda^2 + \lambda)e^{\lambda x} = 0$ and thus the characteristic equation $\lambda(\lambda+1) = 0$ which has roots 0 and -1. Thus $y_h = A + Be^{-x}$ where A and B are arbitrary constants.

To find now the particular solution y_p corresponding to the RHS $r(x)$, note that we have already found a Fourier cosine series expansion for $r(x)$, i.e.,

$$r(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \quad 0 \le x \le 2$$

where

$$\begin{aligned} a_0 &= 1 \\ a_n &= \frac{4}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right) \end{aligned}$$

The particular solution will take the form

$$y_p = y_0 + \sum_{n=1}^{\infty} y_n$$

where y_0 is the particular solution corresponding to the constant term $\frac{a_0}{2}$ and y_n is the particular solution corresponding to $a_n \cos\frac{n\pi x}{2}$ for each n.

Note that y_0 would usually be a constant (as a_0 is) but, as one of the solutions of the homogeneous equations is constant, in this case

$$y_0 = c_0 x$$

for some constant c_0. Substituting into the equation

$$y_0'' + y_0' = \frac{a_0}{2}$$

we obtain

$$c_0 = \frac{a_0}{2} = \frac{1}{2}$$

Similarly y_n satisfies the equation

$$y_n'' + y_n' = a_n \cos\frac{n\pi x}{2}$$

y_n takes the form

$$y_n = A_n \cos\frac{n\pi x}{2} + B_n \sin\frac{n\pi x}{2}$$

and has derivatives

$$\begin{aligned} y_n' &= -\frac{n\pi}{2} A_n \sin\frac{n\pi x}{2} + \frac{n\pi}{2} B_n \cos\frac{n\pi x}{2} \\ y_n'' &= -\left(\frac{n\pi}{2}\right)^2 A_n \cos\frac{n\pi x}{2} - \left(\frac{n\pi}{2}\right)^2 B_n \sin\frac{n\pi x}{2} \end{aligned}$$

Substituting for y_n'' and y_n' then gives

$$\left[-\left(\frac{n\pi}{2}\right)^2 A_n + \frac{n\pi}{2} B_n\right] \cos\frac{n\pi x}{2} - \left[\left(\frac{n\pi}{2}\right)^2 B_n + \frac{n\pi}{2} A_n\right] \sin\frac{n\pi x}{2} = a_n \cos\frac{n\pi x}{2}$$

and equating the coefficients of $\cos\frac{n\pi x}{2}$ and $\sin\frac{n\pi x}{2}$ on each side of the equation we obtain

$$\begin{aligned} -\left(\frac{n\pi}{2}\right)^2 A_n + \frac{n\pi}{2} B_n &= a_n \\ \left(\frac{n\pi}{2}\right)^2 B_n + \frac{n\pi}{2} A_n &= 0 \end{aligned}$$

The second equation gives

$$A_n = -\frac{n\pi}{2} B_n$$

and substituting in the first equation we obtain

$$\left[\left(\frac{n\pi}{2}\right)^3 + \frac{n\pi}{2}\right] B_n = a_n$$

or

$$B_n = \frac{a_n}{\left[\left(\frac{n\pi}{2}\right)^2 + 1\right] \frac{n\pi}{2}}$$

and consequently

$$A_n = -\frac{a_n}{\left[\left(\frac{n\pi}{2}\right)^2 + 1\right]}$$

Thus, the general solution of the differential equation is

$$y = A + Be^{-x} + \frac{x}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\frac{n\pi x}{2} + B_n \sin\frac{n\pi x}{2}\right]$$

with A_n and B_n as above.

Example. RLC circuit

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = \frac{dv(t)}{dt}$$

where $L = 10$ henrys, $R = 100$ ohms, $C = 0.01$ farads; i.e.,

$$10\frac{d^2 i}{dt^2} + 100\frac{di}{dt} + 100i = \frac{dv(t)}{dt}$$

We first find the solution i_h of the homogeneous equation

$$10\frac{d^2 i}{dt^2} + 100\frac{di}{dt} + 100i = 0$$

by setting $i = e^{\lambda t}$ and substituting to obtain the charactertistic equation

$$10\lambda^2 + 100\lambda + 100 = 0$$

which has two (negative) roots $\lambda_{1,2} = -5 \pm \sqrt{15}$ Thus $i_h = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \to 0$ as $t \to \infty$.

Now, consider the case where $v(t) = 100t(\pi^2 - t^2)$ volts for $-\pi < t < \pi$ and let v have period 2π.

We now can find a particular integral i_p of the differential equation by expanding $v'(t) = \frac{dv(t)}{dt}$ in a Fourier series and then finding a particular solution corresponding to each term of the series.

$$v'(t) = 100(\pi^2 - 3t^2)$$

and since $v'(-t) = v'(t)$, $v'(t)$ is even and so has a Fourier cosine series:

$$v'(t) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nt$$

where

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi v'(t)\,dt \\
&= \frac{2}{\pi}\int_0^\pi 100(\pi^2 - 3t^2)\,dt \\
&= \frac{200}{\pi}\left[\pi^2 t - t^3\right]_0^\pi \\
&= \frac{200}{\pi}\left(\pi^3 - \pi^3\right) \\
&= 0
\end{aligned}$$

Or we could simply note that

$$\int_0^\pi v'(t)\,dt = [v(t)]_0^\pi = v(\pi) - v(0) = 0$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi v'(t)\,dt \\
&= \frac{2}{\pi}\left\{\int_0^\pi 100\pi^2 \cos nt\,dt - \int_0^\pi 300t^2 \cos nt\,dt\right\} \\
&= \frac{2}{\pi}\left\{\frac{100\pi^2}{n}\,[\sin nt]_0^\pi - 300\left(\left[\frac{t^2}{n}\sin nt\right]_0^\pi - \frac{2}{n}\int_0^\pi t\sin nt\,dt\right)\right\} \\
&= \frac{1200}{n\pi}\int_0^\pi t\sin nt\,dt \\
&= \frac{1200}{n\pi}\left\{-\left[\frac{t}{n}\cos nt\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos nt\,dt\right\} \\
&= -\frac{1200}{n^2}\cos n\pi + \frac{1200}{n^3\pi}\,[\sin nt]_0^\pi \\
&= (-1)^{n+1}\frac{1200}{n^2}
\end{aligned}$$

That is

$$v'(t) = 1200\left(\cos t - \frac{1}{4}\cos 2t + \frac{1}{9}\cos 3t - \dots\right)$$

Let i_n be the particular integral corresponding to $\dfrac{\cos nt}{n^2}$.This takes the form

$$i_n = A_n \cos nt + B_n \sin nt$$

where A_n and B_n are constants (which depend on n)

$$i_n' = -nA_n \sin nt + nB_n \cos nt$$

and

$$i_n'' = -n^2 A_n \cos nt - n^2 B_n \sin nt$$

Substituting into the equation for i_n, i.e.,

$$10\frac{d^2 i_n}{dt^2} + 100\frac{di_n}{dt} + 100 i_n = \frac{\cos nt}{n^2}$$

gives

$$(-10n^2A_n+100nB_n+100A_n)\cos nt+(-10n^2B_n-100nA_n+100B_n)\sin nt = \frac{\cos nt}{n^2}$$

Equating the coefficients of $\cos nt$ and $\sin nt$ on each side of this equation (and dividing by 10) gives

$$(10-n^2)B_n - 10nA_n = 0$$

$$(10-n^2)A_n + 10nB_n = \frac{1}{10n^2}$$

Thus the first equation $\Rightarrow$

$$A_n = \frac{10-n^2}{10n}B_n$$

and substituting this for A_n in the second equation gives

$$\left[\frac{(10-n^2)^2}{10n} + 10n\right]B_n = \frac{1}{10n^2}$$

or

$$\left[\frac{(10-n^2)^2+100n^2}{10n}\right]B_n = \frac{1}{10n^2}$$

and so

$$B_n = \frac{1}{n\left[(10-n^2)^2+100n^2\right]}$$

and consequently

$$A_n = \frac{10-n^2}{10n^2\left[(10-n^2)^2+100n^2\right]}$$

Therefore the particular integral for $v'(t)$ is

$$i_p = 1200\sum_{n=1}^{\infty}(-1)^{n+1}\left[A_n\cos nt + B_n\sin nt\right]$$

with A_n and B_n defined as above.