3.1 The Concept of a Game

A game is defined by

a) The set of players (at least two "rational" players).

b) The set of actions available to each player (each has at least two possible actions).

c) A payoff function, which gives the (expected) payoff of each player given the actions used by the players. The (expected) payoff of a player does depend on the actions used by the other players.

3.1 The Concept of a Game

According to such a definition, roulette is not a mathematical game, since the expected winnings of a player only depend on how he/she plays.

The lottery is a mathematical game. Although the probability of winning the jackpot is independent of the choices of other players, a player can maximise his/her expected reward by choosing combinations of numbers that other individuals do not choose (such an individual is not more likely to win the jackpot, but when he/she wins, he/she wins more on average).

3.2 The Matrix Form of a 2-Player Game

Assume that each player has a finite set of actions to choose from.

In the matrix form of a 2-player game, each row corresponds to an action of Player 1 and each column corresponds to an action of Player 2.

Each cell of the payoff matrix is associated with a payoff vector. The i-th component of this vector gives the payoff to Player i.

3.2 The Matrix Form of a 2-Player Game

For example, the following describes a so called Hawk-Dove game

	H	D
H	(-2,-2)	(4,0)
D	(0,4)	(2,2)

For example, when Player 1 plays H and Player 2 plays D, Player 1 obtains a payoff of 4 and Player 2 obtains a payoff of 0.

3.2 The Matrix Form of a 2-Player Game

In general, a 2-player matrix game can be described by

1. The set of actions available to Player 1, $A = \{a_1, a_2, \ldots, a_m\}$.
2. The set of actions available to Player 2, $B = \{b_1, b_2, \ldots, b_n\}$.
3. The $m \times n$ matrix of 2-dimensional vectors giving the payoffs of both players for each of the $m \times n$ possible combinations of actions. The payoff of Player k when Player 1 plays a_i and Player 2 plays b_j will be denoted $R_k(a_i, b_j)$.

3.2 The Matrix Form of a 2-Player Game

One advantage of the matrix form of a game is its simplicity.

One disadvantage is that it is assumed that players choose their actions simultaneously. That is to say, the actions may be interpreted as strategies which are chosen at the beginning of a game.

As we continue through this chapter, we will differentiate between the concept of action and the concept of strategy.

Zero-sum and Constant-sum Games

A game is said to be a constant sum game, if the sum of the payoffs obtained by the players is fixed, regardless of the combination of strategies used.

A game is said to be a zero-sum game, if the sum of payoffs obtained by the players is always 0, regardless of the combination of strategies used.

A game of constant sum k is essentially the same as a zero-sum game. A referee could pay both of the players $k/2$ and require that they play the game in which each payoff is reduced by $k/2$ (this would be a zero-sum game).

2-player constant-sum games are games of **pure conflict**. Whatever one player gains, the other will lose.

3.3 The Extensive Form of a Game

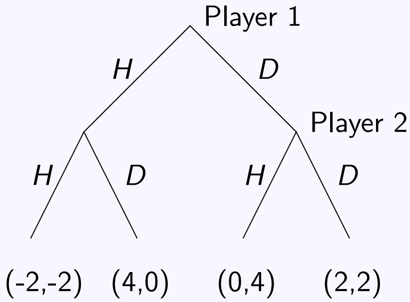
In many games, e.g. chess, each player makes a sequence of moves.

Such a game can be represented by a tree. Each node represents a position (state) in which a player must make a move.

Each edge coming out of a node represents a move that the player can make in that particular state.

Each terminal point of the tree is associated with a payoff vector.

The Asymmetric Hawk-Dove Game



The Asymmetric Hawk-Dove Game

In such a game, Player 1 first decides whether to play H and D.

After observing the decision of Player 1, Player 2 then decides whether to play H and D.

For example, if Player 1 chooses H and Player 2 chooses D, Player 1 obtains a payoff of 4 and Player 2 obtains a payoff of 0.

The Asymmetric Hawk-Dove Game

It might appear that this game is equivalent to the matrix form of the Hawk-Dove given above.

In order to see the difference between these games, we should differentiate between strategies and actions.

In a game represented in extensive form, a strategy is a rule that defines which action a player should play at all the nodes where he/she makes a move.

An action is the observed behaviour resulting from such a strategy.

The Asymmetric Hawk-Dove Game

Here Player 1 first chooses between H and D. This is her only choice. Thus Player 1's strategies correspond exactly to her possible actions (H and D).

However, since Player 2 observes the move made by Player 1, he can condition his move on the move made by Player 1.

For each of the moves made by Player 1, Player 2 has 2 possible moves. Hence, he has $2 \times 2 = 4$ possible strategies. These are listed on the following slide.

The Asymmetric Hawk-Dove Game

In the extended game described above, Player 2's strategies are

1. $H|H, H|D$, i.e. play H when Player 1 plays H and play H when Player 1 plays D, in other words always play H.
2. $H|H, D|D$, i.e. take the same action as Player 1.
3. $D|H, D|D$, i.e. always play D.
4. $D|H, H|D$, i.e. play D when Player 1 plays H and play H when Player 1 plays D.

In the matrix game presented earlier, Player 2 cannot condition his action on the action made by Player 1. Hence, he only has 2 strategies corresponding to the two actions H and D.

Simultaneous Moves in Extensive Form Games

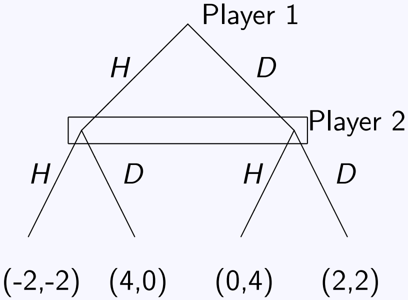
Although the extensive form of the game is designed to describe games in which a sequence of moves are made, they can be adapted to describe games in which moves are made simultaneously.

This is done using information sets. Suppose two nodes represent states in which one player moves and that player cannot differentiate between the states.

These two nodes are said to be in the same **information set**. An information set is represented by a box.

A player may only condition an action on the information set, he/she is presently in.

The Symmetric Hawk-Dove Game in Extended Form



In this case both players have 2 possible strategies: H and D.

Translating from Extensive Form into Matrix Form

In order to translate the description of a game from extensive form to matrix form, we

1. List the possible strategies of both players.
2. In the matrix describing the game, the rows correspond to the strategies of Player 1 and the columns correspond to the strategies of Player 2.
3. We then derive the payoff vector corresponding to each possible strategy pair.

Translating from Extensive Form into Matrix Form

It should be noted that each game in extensive form has a unique definition as a matrix game (apart from possible differences in the labelling of the strategies).

However, there may be various games in extensive form corresponding to a game given in matrix form. Thus it is generally not possible to translate a game in matrix form into a game in extensive form.

This is unsurprising, since the extensive form of a game gives more detailed description of how the game is actually played.

Translating the Asymmetric Hawk-Dove Game into Matrix Form

Since Player 1 has 2 strategies and Player 2 has 4 strategies, the payoff matrix will be of dimension 2×4.

Player 1 can play H or D.

Player 2 can play $(H|H, H|D)$, $(H|H, D|D)$, $(D|H, D|D)$ or $(D|H, H|D)$.

Translating the Asymmetric Hawk-Dove Game into Matrix Form

We now consider the payoff vectors associated with each strategy pair.

Suppose Player 1 plays H.

a) If Player 2 plays $(H|H, H|D)$ or $(H|H, D|D)$, then both players take the action H and the resulting payoff vector is $(-2, -2)$.

b) If Player 2 plays $(D|H, D|D)$ or $(D|H, H|D)$, then Player 1 takes the action H and Player 2 takes the action D and the resulting payoff vector is $(4, 0)$.

Translating the Asymmetric Hawk-Dove Game into Matrix Form

Now suppose Player 1 plays D.

a) If Player 2 plays $(D|H, D|D)$ or $(H|H, D|D)$, then both players take the actions D and the resulting payoff vector is $(2, 2)$.

b) If Player 2 plays $(H|H, H|D)$ or $(D|H, H|D)$, then Player 1 takes the action D and Player 2 takes the action H and the resulting payoff vector is $(0, 4)$.

Translating the Asymmetric Hawk-Dove Game into Matrix Form

It follows that the matrix form of the asymmetric game is given by

	$(H\|H, H\|D)$	$(H\|H, D\|D)$	$(D\|H, D\|D)$	$(D\|H, H\|D)$
H	(-2,-2)	(-2,-2)	(4,0)	(4,0)
D	(0,4)	(2,2)	(2,2)	(0,4)

Moves by "Nature"

Extensive form can also be used to describe moves by "nature", i.e. random events, the roll of a die, dealing cards.

Whenever nature is called to make a move at a given node, the edges from this node correspond to the possible results. The probability of each result should also be given.

When such a game is written in matrix form, we only consider the strategies that the players can use. In order to define the vector of **expected** payoffs given the combination of strategies used, we take expectations with respect to the moves of nature (i.e. nature is assumed to be "random" or "non-rational").

Example

Suppose that player 1 can first choose either A or B. Player 2 does not know this choice.

Afterwards Player 2 observes the result of a coin toss. Regardless of the result of the coin toss, Player 2 can play A or B.

If the coin toss results in heads, when both players choose the same action Player 1 obtains \$1 from Player 2, but gives Player 2 \$1 if each chooses different actions. If the coin toss results in tails, when each player chooses different actions Player 1 obtains \$2 from Player 2, otherwise he gives Player 2 \$2 (note this is a zero-sum game).

Example

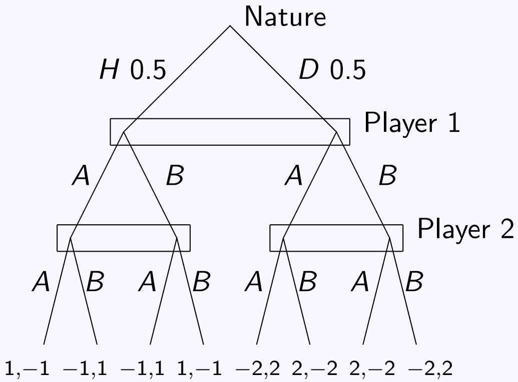
When writing the game in extensive form, we can often shift a move to an different position in the tree than its natural position, in order to make a more legible tree.

The important thing here is to correctly describe what information each player has when making a move.

In this example, Player 1 does not know the result of the coin toss (or Player 2's move).

Player 2 does not know Player 1's move, but knows the result of the coin toss. Hence, Player 2's move must come lower down the tree than the coin toss. We can order the moves as follows: nature, Player 1, Player 2.

Example


Nature
H 0.5
D 0.5
Player 1
A
B
A
B
Player 2
A
B
A
B
A
B
A
B
1,−1
−1,1
−1,1
1,−1
−2,2
2,−2
2,−2
−2,2

Example

In order to describe the game in matrix form, note that Player 1 has no information regarding the coin toss or Player 2's move. It follows that she has 2 strategies: A and B.

Player 2 has no information regarding Player 1's move, but knows the result of the coin toss. Hence, his action can be made conditional on the result of the coin toss. He has 4 strategies: $(A|H, A|T)$, $(A|H, B|T)$, $(B|H, B|T)$, $(B|H, A|T)$.

Example

Suppose Player 1 plays A.

When Player 2 plays $(A|H, A|T)$, i.e. always A, Player 1 wins 1 if the result of the coin toss was H and loses 2 if the result of the coin toss was tails. Player 1's expected reward is $0.5 \times 1 - 0.5 \times 2 = -0.5$. Since the game is zero-sum, Player 2's expected payoff is 0.5.

When Player 2 plays $(B|H, B|T)$, i.e. always B, Player 1 loses 1 if the result of the coin toss was H and wins 2 if the result of the coin toss was tails. Player 1's expected reward is $0.5 \times 2 - 0.5 \times 1 = 0.5$.

The expected rewards under all the other possible pairs of strategies can be calculated in a similar way.

Example

The matrix form of this game is given by

	$(A\|H, A\|T)$	$(A\|H, B\|T)$	$(B\|H, B\|T)$	$(B\|H, A\|T)$
A	(-0.5,0.5)	(1.5,-1.5)	(0.5,-0.5)	(-1.5,1.5)
B	(0.5,-0.5)	(-1.5,1.5)	(-0.5,0.5)	(1.5,-1.5)

Consider the following game:

Player 1 chooses a number x between 0 and 1. Having observed the choice of Player 1, Player 2 chooses a number y between 0 and 1.

The payoff of Player 1 is given by $R_1(x, y) = 4xy - x$. The payoff of Player 2 is given by $R_2(x, y) = 4xy - y$.

In order to depict choice from a continuum, instead of using branches we can use a triangle with a horizontal base. The strategy set is described alongside the corresponding triangle.

Extensive Forms of Games with a Continuum of Strategies

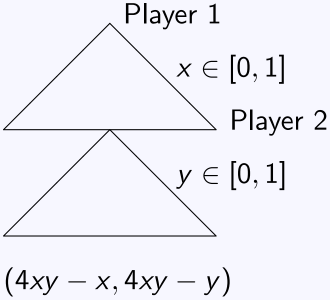
If a move is unobserved by the next player to move, the base of the triangle should be enclosed in a box denoting an information set.

Suppose the second player can observe whether the move of Player 1 belongs to certain intervals. Each of these intervals corresponds to an information set.

The possible moves of Player 2 corresponding to each information set should be depicted by triangles extending out of these sets.

Example 3.3.1

The extensive form of the game described above is



The Concepts of Complete Information and Perfect Information

A game is a game with **complete information** if both players know both the actions available to the other players and the payoffs obtained by other players under all the possible combinations of strategies.

In addition, if each player always knows which node of the extensive form of the game he/she is at when he/she has to make a move, then such a game is a game with **perfect information**.

The Concepts of Complete Information and Perfect Information

For example, chess is a game of perfect information (given a payoff structure of 1 point for a win, 0.5 for a draw, 0 for a loss), since the state of the game is always known, each player knows which moves are available to the opponent.

Bridge is definitely not a game of perfect information, as e.g. the bidder does not know how the cards are split between his opponents. However, it may be argued that it is a game of complete information, since the scoring system is well defined and players know what others are allowed to play given their hand and the bidding (a full description of a strategy would however be very complex).

Solution of Games with Perfect Information

In games of perfect information, moves are made in succession and each of the previous moves are known to each player.

Such games can be solved by recursion based on the extensive form of the game.

Consider the asymmetric hawk-dove game given above. The final move is made by Player 2. Given the move made by Player 1 (H or D), Player 2 simply maximises his expected reward.

This defines Player 2's optimal action after H and his optimal action after D.

Solution of Games with Perfect Information

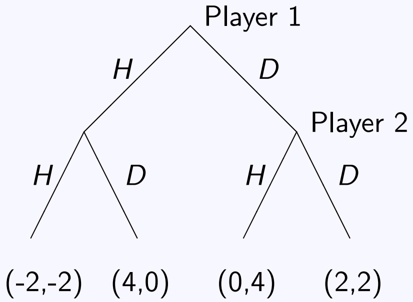
Working backwards, Player 1 assumes that Player 2 will use his best response to her action.

This defines the action that Player 1 should take.

It follows that there always exists a solution to a game with perfect information. Unless at any node a player is indifferent between the actions he/she can take, this solution will be unique.

Any vector of payoffs corresponding to the solution of such a game is called a **value** of the game.

The Asymmetric Hawk-Dove Game



Solution of Games with Perfect Information

Suppose Player 1 has played H. Player 2 obtains -2 by playing H and 0 by playing D. Hence, his best response to H is to play D.

Suppose Player 1 has played D. Player 2 obtains 4 by playing H and 2 by playing D. Hence, his best response to D is H.

Now we consider the action of Player 1.

Solution of Games with Perfect Information

If she plays H, then Player 2 will respond by playing D. Player 1 obtains a reward of 4.

If she plays D, then Player 2 will respond by playing H. Player 1 obtains a reward of 0.

It follows that Player 1 should play H. Player 2 follows by playing D.

The value of the game is $(4, 0)$.

Equilibrium Path and Subgame Perfect Equilibria

The set of actions observed at a solution of an extensive form game with perfect information is called an **equilibrium path**.

In the Hawk-Dove game considered above, the equilibrium path is (H, D).

However, such a path does not describe how players should react to "mistakes", i.e. how should individuals act off the equilibrium path.

An equilibrium is said to be **subgame perfect**, if starting from any node on the game tree, players play an equilibrium pair of strategies, i.e. an equilibrium is played in any subgame of the game in question.

Equilibrium Path and Subgame Perfect Equilibria

In this case, the subgame perfect equilibrium is defined by giving the optimal response of Player 2 to each action of Player 1 and the optimal action of Player 1.

These were derived during the recursive solution of the game.

Player 2 should respond to H by playing D and respond to D by playing H. Hence, Player 2's subgame perfect equilibrium strategy is $(D|H, H|D)$.

The subgame perfect equilibrium strategy of Player 2 is H.

If nature has moves in a game of perfect information, then each player is assumed to maximise his/her expected reward at each stage of the recursion procedure.

Concepts of Pure and Mixed Strategies

In games of perfect information, it is clear that unless a player is indifferent between two actions, then he/she should never randomise.

However, in games of imperfect information (e.g. when moves are made simultaneously like rock-scissors-paper) it is clear that one player may not want the other to "guess" which action he/she is going to take.

In such cases individuals choose the action they take at random, i.e. they use a **mixed strategy**.

Concepts of Pure and Mixed Strategies

The matrix form of the rock-scissors-paper game is:

	R	S	P
R	(0,0)	(1,-1)	(-1,1)
S	(-1,1)	(0,0)	(1,-1)
P	(1,-1)	(-1,1)	(0,0)

Intuitively, at equilibrium both players choose each action with a probability of $1/3$ (see tutorial sheet for details).

Concepts of Pure and Mixed Strategies

If a player always chooses the same action in a matrix game, then he/she is using a so called **pure strategy**.

It is normally assumed that players choose their actions independently of each other (i.e. there is no communication).

Later we will consider games in which players can communicate, i.e. the actions they take may be correlated.

Concepts of Pure and Mixed Strategies

Suppose that in the rock-scissors-paper game, Player 1 plays rock, scissors and paper with probability p_R, p_S and p_P, respectively. Player 2 plays rock, scissors and paper with probability q_R, q_S and q_P. The probability distribution over the set of strategy pairs is

	R	S	P
R	$p_R q_R$	$p_R q_S$	$p_R q_P$
S	$p_S q_R$	$p_S q_S$	$p_S q_P$
P	$p_P q_R$	$p_P q_S$	$p_P q_P$

Concepts of Pure and Mixed Strategies

When players use mixed strategies, the expected rewards of the players can be calculated by taking expectations with respect to the probability distribution over the set of strategy pairs. Hence,

$$\begin{aligned} R_1(M_1, M_2) =& p_R q_R R_1(R, R) + p_R q_S R_1(R, S) + p_R q_P R_1(R, P) + \\ &+ p_S q_R R_1(S, R) + p_S q_S R_1(S, S) + p_S q_P R_1(S, P) + \\ &+ p_P q_R R_1(P, R) + p_P q_S R_1(P, S) + p_P q_P R_1(P, P) \\ =& p_R q_S + p_S q_P + p_P q_R - p_S q_R - p_P q_S - p_R q_P. \end{aligned}$$