# Introduction to Geometric Control

## 1 Relative Degree

Consider the square (no. of inputs = no. of outputs = $p$) affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} = \mathbf{f}(\mathbf{x}) + \begin{bmatrix}\mathbf{g}_1(\mathbf{x}), & \ldots, & \mathbf{g}_p(\mathbf{x})\end{bmatrix}\mathbf{u} \tag{1}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) = \begin{bmatrix}\mathbf{h}_1(\mathbf{x}) \\ \vdots \\ \mathbf{h}_p(\mathbf{x})\end{bmatrix} \tag{2}$$

(This system is assumed to have an equilibrium or operating point corresponding to $\mathbf{u}_e \equiv \mathbf{0}$ at $\mathbf{x}_e = \mathbf{0}$ with $\mathbf{y}_e = \mathbf{0}$.)
Recall that for the $i$-th output $y_i$, the relative degree $r_i$ is the first time - derivative of $y_i$ for which $\mathbf{u}$ appears in the expression for the derivative.

### 1.1 SISO

For example, for a SISO system, this translates to

$$L_{\mathbf{g}}L_{\mathbf{f}^{k-1}}h(\mathbf{x}) = 0 \Leftrightarrow \frac{d^k y}{dt^k} = L_{\mathbf{f}^k}h(\mathbf{x}), \qquad k = 1, 2, \ldots, r-1 \tag{3}$$

$$L_{\mathbf{g}}L_{\mathbf{f}^{r-1}}h(\mathbf{x}_e) \neq 0 \Leftrightarrow \frac{d^r y}{dt^r} = L_{\mathbf{f}^r}h(\mathbf{x}) + L_{\mathbf{g}}L_{\mathbf{f}^{r-1}}h(\mathbf{x})u \tag{4}$$

For nonlinear systems, the relative degree is a local concept, defined in some neighborhood of $\mathbf{x} = \mathbf{x}_e$. If conditions (3) and (4) hold globally, we say that the system has a global relative degree r.

### 1.2 MIMO

The MIMO system (1)-(2) has relative degree $\{r_1, r_2, \ldots, r_p\}$ at $\mathbf{x} = \mathbf{x}_e$ if

- $$\begin{aligned} L_{\mathbf{g}_j}L_{\mathbf{f}^{k-1}}h_i(\mathbf{x}) &= 0, \qquad 1 \leq i, j \leq p, \;\; k = 1, 2, \ldots, r_i - 1 \\ \Leftrightarrow \frac{d^k y_i}{dt^k} &= L_{\mathbf{f}^k}h_i(\mathbf{x}), \qquad 1 \leq i \leq p, \;\; k = 1, 2, \ldots, r-1 \end{aligned} \tag{5}$$

for all $\mathbf{x}$ in a neighbourhood of $\mathbf{x} = \mathbf{x}_e$.

- The $p \times p$ matrix $N(\mathbf{x})$ is invertible at $\mathbf{x} = \mathbf{x}_e$ where

$$\begin{aligned} N(\mathbf{x}) = \left[\frac{\partial y_i^{(r_i)}}{\partial u_j}\right]_{i,j} &= \begin{bmatrix} L_{\mathbf{g}_1} L_{\mathbf{f}^{r_1-1}} h_1(\mathbf{x}) & \cdots & L_{\mathbf{g}_p} L_{\mathbf{f}^{r_1-1}} h_1(\mathbf{x}) \\ \vdots & & \vdots \\ L_{\mathbf{g}_1} L_{\mathbf{f}^{r_p-1}} h_p(\mathbf{x}) & \cdots & L_{\mathbf{g}_p} L_{\mathbf{f}^{r_p-1}} h_p(\mathbf{x}) \end{bmatrix} \\ \Leftrightarrow \begin{pmatrix} \frac{d^{r_1} y_1}{dt^{r_1}} \\ \vdots \\ \frac{d^{r_p} y_p}{dt^{r_p}} \end{pmatrix} &= \underbrace{\begin{bmatrix} L_{\mathbf{f}^{r_1}} h_1(\mathbf{x}) \\ \vdots \\ L_{\mathbf{f}^{r_p}} h_p(\mathbf{x}) \end{bmatrix}}_{G(\mathbf{x})} + N(\mathbf{x})\mathbf{u} \end{aligned} \tag{6}$$

Condition (6) is the MIMO generalisation of the condition (4) in the SISO case. If $r_1 = r_2 = \cdots = r_p$, we say that the system has a uniform relative degree $r_1$.

# 2 Normal Form

## 2.1 SISO

For a SISO system of relative degree $r$, consider the state transformation

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{z} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad \mathbf{\Phi}(\mathbf{x}_e) = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

where $\xi \in \mathbb{R}^r$, $\eta \in \mathbb{R}^{n-r}$, defined in part as follows:

$$\begin{aligned} \xi_1 = \mathbf{\Phi}_1(\mathbf{x}) \stackrel{\triangle}{=} & \; y & = & \; h(\mathbf{x}) \\ \xi_2 = \mathbf{\Phi}_2(\mathbf{x}) \stackrel{\triangle}{=} & \; \dot{y} & = & \; L_{\mathbf{f}} h(\mathbf{x}) \\ \vdots & & \vdots & \\ \xi_r = \mathbf{\Phi}_r(\mathbf{x}) \stackrel{\triangle}{=} & \; y^{(r-1)} & = & \; L_{\mathbf{f}^{r-1}} h(\mathbf{x}) \end{aligned} \tag{7}$$

Since condition (4) holds, this immediately gives that

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= \underbrace{L_{\mathbf{f}^r} h(\mathbf{x})}_{A(\mathbf{x})} + \underbrace{L_{\mathbf{g}} L_{\mathbf{f}^{r-1}} h(\mathbf{x})}_{B(\mathbf{x})} u
\end{aligned} \tag{8}
$$

It may be shown that the following holds.
**Lemma 1** (Linear independence of output derivatives):
The row vectors $\{\frac{\partial \mathbf{\Phi}_1}{\partial \mathbf{x}}(\mathbf{x}_e), \frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}(\mathbf{x}_e), \ldots, \frac{\partial \mathbf{\Phi}_r}{\partial \mathbf{x}}(\mathbf{x}_e)\}$ are linearly independent.

Hence, if $r = n$, then $\mathbf{\Phi}(\mathbf{x})$ is indeed a well-defined transformation in a neighbourhood of $\mathbf{x} = \mathbf{x}_e$ since its Jacobian is invertible.

On the other hand, if $r < n$, it is always possible to choose $\mathbf{\Phi}_{r+1}(\mathbf{x}), \mathbf{\Phi}_{r+2}(\mathbf{x}), \ldots, \mathbf{\Phi}_n(\mathbf{x})$ so that $\mathbf{\Phi}(\mathbf{x})$ has a non-singular Jacobian at $\mathbf{x}_e$ and so is indeed a well-defined transformation in a neighbourhood of $\mathbf{x} = \mathbf{x}_e$. Hence the dynamics of the remaining transformed state variables $\eta$ may be described by

$$
\begin{aligned}
\eta_i &= \mathbf{\Phi}_i(\mathbf{x}), \qquad i = r+1, r+2, \ldots, n \\
\Rightarrow \dot{\eta}_i &= \frac{\partial \mathbf{\Phi}_i}{\partial \mathbf{x}} \left(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) u\right), \qquad i = r+1, r+2, \ldots, n \\
&= \underbrace{L_{\mathbf{f}} \mathbf{\Phi}_i}_{D_i}(\mathbf{x}) + \underbrace{L_{\mathbf{g}} \mathbf{\Phi}_i}_{E_i}(\mathbf{x}) u, \qquad i = r+1, r+2, \ldots, n
\end{aligned} \tag{9}
$$

Moreover it is always possible to choose $\mathbf{\Phi}_{r+1}(\mathbf{x}), \mathbf{\Phi}_{r+2}(\mathbf{x}), \ldots, \mathbf{\Phi}_n(\mathbf{x})$ so that
**Lemma 2** (Non-dependence of "remnant states" on the input):

$$
L_{\mathbf{g}} \mathbf{\Phi}_i(\mathbf{x}) = 0, \qquad i = r+1, r+2, \ldots, n
$$

for all $\mathbf{x}$ in a neighbourhood of $\mathbf{x}_e$.
And so Eq (9 ) becomes

$$
\dot{\eta}_i = D_i(\mathbf{x}), \qquad i = r+1, r+2, \ldots, n \tag{10}
$$

Using the fact $\mathbf{x} = \mathbf{\Phi}^{-1}(\mathbf{z})$, the state equations (Eq (8) and Eq (10) ) and output equation may be written in *normal form*:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= A(\mathbf{\Phi}^{-1}(\mathbf{z})) + B(\mathbf{\Phi}^{-1}(\mathbf{z}))u \\
\dot{\eta}_{r+1} &= D_{r+1}(\mathbf{\Phi}^{-1}(\mathbf{z})) \\
\dot{\eta}_{r+2} &= D_{r+2}(\mathbf{\Phi}^{-1}(\mathbf{z})) \\
&\vdots \\
\dot{\eta}_n &= D_n(\mathbf{\Phi}^{-1}(\mathbf{z})) \\
y &= \xi_1
\end{aligned} \tag{11}$$

**Remark**: Though Lemma 2 holds in a theoretical sense, it may be difficult to find the required transformation. In which case, the state and output equations (not in normal form) are

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= A(\mathbf{\Phi}^{-1}(\mathbf{z})) + B(\mathbf{\Phi}^{-1}(\mathbf{z}))u \\
\dot{\eta}_{r+1} &= D_{r+1}(\mathbf{\Phi}^{-1}(\mathbf{z})) + E_{r+1}(\mathbf{\Phi}^{-1}(\mathbf{z}))u \\
\dot{\eta}_{r+2} &= D_{r+2}(\mathbf{\Phi}^{-1}(\mathbf{z})) + E_{r+2}(\mathbf{\Phi}^{-1}(\mathbf{z}))u \\
&\vdots \\
\dot{\eta}_n &= D_n(\mathbf{\Phi}^{-1}(\mathbf{z})) + E_n(\mathbf{\Phi}^{-1}(\mathbf{z}))u \\
y &= \xi_1
\end{aligned} \tag{12}$$

## 3 I/O Feedback Linearisation

### 3.1 SISO

By definition of relative degree $B(.) \neq 0$ (see Eq (11)). Hence by using the "cancellation" feedback control

$$\begin{aligned} A(\mathbf{\Phi}^{-1}(\mathbf{z})) + B(\mathbf{\Phi}^{-1}(\mathbf{z}))u &= v \\ \Rightarrow u &= \frac{-A(\mathbf{\Phi}^{-1}(\mathbf{z})) + v}{B(\mathbf{\Phi}^{-1}(\mathbf{z}))} \end{aligned} \tag{13}$$

the $\xi$-subsystem can be converted into a completely controllable linear subsystem consisting of $r$ cascaded integrators driven by input $v$, which then can have its dynamics redesigned by e.g. pole placement or LQ methods etc. In normal form, the $\eta$ -subsystem is not observable; it represents the "remnant", and so must have stable dynamics for the approach to be useful. This is investigated in the section entitled **Zero Dynamics**.

## 4 Zero Dynamics

### 4.1 SISO

Keeping $y(t) \equiv 0$ ("zeroing the output") requires that $\xi_1(t) \equiv 0$ (see output equation) leading to $\xi(t) \equiv \mathbf{0}$. Thus $\xi_r(t) \equiv 0$ in turn requires that $v \equiv 0$ and so $u$ must be chosen as

$$u = u_{zd} \stackrel{\triangle}{=} -\frac{A(\mathbf{\Phi}^{-1}(\mathbf{0}, \eta))}{B(\mathbf{\Phi}^{-1}(\mathbf{0}, \eta))}.$$

The remnant or zero dynamics are then

$$\dot{\eta} = \underbrace{\mathbf{D}(\mathbf{\Phi}^{-1}(\mathbf{0}, \eta)) + \mathbf{E}(\mathbf{\Phi}^{-1}(\mathbf{0}, \eta))u_{zd}}_{\mathbf{F}(\eta)} \tag{14}$$

The stability of the zero (equilibrium) solution of this flow can be tested by any of the usual methods.

# 5 Examples

## 5.1 Example 1

Consider the 2nd order affine system

$$
\begin{aligned}
\dot{\mathbf{x}} &= \underbrace{\begin{pmatrix} x_2 + 4x_1^2 x_2 \\ -x_1 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} u \\
y &= \underbrace{x_1}_{h(\mathbf{x})}
\end{aligned}
$$

**Computing the relative degree**

$$
\begin{aligned}
y &= x_1 \\
\Rightarrow \dot{y} &= \dot{x}_1 = x_2(1 + 4x_1^2) \\
\Rightarrow \ddot{y} &= \dot{x}_2(1 + 4x_1^2) + x_2(8x_1\dot{x}_1) = (-x_1 + u)(1 + 4x_1^2) + 8x_1x_2^2(1 + 4x_1^2) \\
&= \underbrace{x_1(1 + 4x_1^2)(-1 + 8x_2^2)}_{L_{\mathbf{f}^2}h(x_1,x_2)=A(x_1,x_2)} + \underbrace{(1 + 4x_1^2)}_{L_{\mathbf{g}}L_{\mathbf{f}}h(x_1,x_2)=B(x_1,x_2)} u
\end{aligned}
$$

Therefore $r = 2$. Hence $r = n$ and so there is no remnant.
**Describing the Normal Form**

$$
\begin{aligned}
\xi_1 = \mathbf{\Phi}_1(\mathbf{x}) = \quad h(\mathbf{x}) &= x_1 \\
\xi_2 = \mathbf{\Phi}_2(\mathbf{x}) = L_{\mathbf{f}}h(\mathbf{x}) &= x_2(1 + 4x_1^2)
\end{aligned}
$$

Solving to get the inverse transformation

$$
\begin{aligned}
\Rightarrow x_1 = \mathbf{\Phi}^{-1}(\xi) &= \xi_1 \\
x_2 = \mathbf{\Phi}^{-1}(\xi) &= \frac{\xi_2}{1 + 4\xi_1^2}
\end{aligned}
$$

The normal form is

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= A + Bu \\
y &= \xi_1
\end{aligned}
$$

**I/O Linearisation Feedback**
With

$$u = \frac{-A+v}{B} = \frac{-x_1(1+4x_1^2)(-1+8x_2^2)+v}{1+4x_1^2} = -x_1(-1+8x_2^2) + \frac{v}{1+4x_1^2}$$

the linearised system is

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v \\ y &= \xi_1\end{aligned}$$

**Stabilising feedback**
This double integrator system can be stabilised by

$$\begin{aligned}v &= k_1\xi_1 + k_2\xi_2 + w \\ &= k_1x_1 + k_2x_2(1+4x_1^2) + w\end{aligned}$$

where $k_1, k_2 < 0$. Hence

$$u = -x_1(-1+8x_2^2) + k_1\frac{x_1}{1+4x_1^2} + k_2x_2 + \frac{w}{1+4x_1^2}$$

## 5.2 Example 2

Consider the 2nd order affine system (same state equation as Ex. 1 but different output).

$$\begin{aligned}\dot{\mathbf{x}} &= \underbrace{\begin{pmatrix} x_2 + 4x_1^2x_2 \\ -x_1 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} u \\ y &= \underbrace{x_2}_{h(\mathbf{x})}\end{aligned}$$

**Computing the relative degree**

$$\begin{aligned}y &= x_1 \\ \Rightarrow \dot{y} &= \dot{x}_2 = \underbrace{-x_1}_{L_{\mathbf{f}}h(x_1,x_2)=A(x_1,x_2)} + \underbrace{1}_{L_{\mathbf{g}}h(x_1,x_2)=B(x_1,x_2)} u\end{aligned}$$

Therefore $r = 1$. Hence $r < n$ and so there is a remnant.
**Describing the Normal Form**

$$\begin{aligned} \xi_1 = \mathbf{\Phi}_1(\mathbf{x}) = \quad h(\mathbf{x}) &= x_2 \\ \eta_2 = \mathbf{\Phi}_2(\mathbf{x}) & \\ \Rightarrow \dot{\eta}_2 = \frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}(\dot{\mathbf{x}}) &= \frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}\left(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u\right) \\ &= \underbrace{\frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})}_{L_\mathbf{f}\mathbf{\Phi}_2(\mathbf{x})} + \underbrace{\frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})}_{L_\mathbf{g}\mathbf{\Phi}_2(\mathbf{x})} u \end{aligned}$$

From Lemma 2 it is known that $\mathbf{\Phi}_2$ can be chosen so that $L_\mathbf{g}\mathbf{\Phi}_2(\mathbf{x}) = 0$, but we calculate that $L_\mathbf{g}\mathbf{\Phi}_2(\mathbf{x}) = \frac{\partial \mathbf{\Phi}_2}{\partial x_2}$ and so $\mathbf{\Phi}_2(\mathbf{x}) = C(x_1)$, where $C$ is a function of its argument. Since $\eta_2 = \mathbf{\Phi}_2(\mathbf{x}) = C(x_1)$, $C$ must also be invertible. Solving to get the inverse transformation

$$\begin{aligned} \Rightarrow x_1 = \mathbf{\Phi}^{-1}(\xi) &= C^{-1}(\eta_2) \\ x_2 = \mathbf{\Phi}^{-1}(\xi) &= \xi_1 \end{aligned}$$

The normal form is

$$\begin{aligned} \dot{\xi}_1 &= A + Bu = -x_1 + u \\ \dot{\eta}_2 &= \frac{dC}{dx_1}(1 + 4x_1^2)x_2 \\ y &= \xi_1 \end{aligned}$$

**(I/O Linearisation Feedback)**

**Zero Dynamics**
For this example

$$y \equiv 0 \Leftrightarrow x_2 \equiv 0.$$

Hence the zero dynamics are (see Normal form above)

$$\dot{\eta}_2 = 0$$

What does this imply about the behaviour of this control system?

# 6 Exact Feedback Linearisation

The previous two examples demonstrate some features of geometric control which will be expanded on in this section: In Example 1, the system has relative degree equal to system order, no remnant, is transformable to a linear integrator-cascade system and is I/O stabilisable. Example 2 has relative degree less than the system order, a remnant whose zero dynamics are marginally stable but which can be I/O stabilised.

**Proposition** There exists an output function $y = h(\mathbf{x})$ for which the relative degree $r$ equals the system order $n$ is necessary and sufficient for the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u$ to have a "linearising" transformation $\mathbf{z} = \mathbf{\Phi}(\mathbf{x})$ and a "linearising" feedback $u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v$ [1] which transforms the system into a completely controllable linear system. The output $y$ may be real or "synthetic".

Example 1 gives a constructive proof method for sufficiency which may be generalised to $n$-th order system.. The proof of necessity needs the following two results.

**Lemma 3** (Invariance of $r$ under state transformation):
Setting

$$\overline{\mathbf{f}}(\mathbf{z}) = \left[\frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})\right]_{\mathbf{x}=\mathbf{\Phi}^{-1}(\mathbf{z})}, \qquad \overline{\mathbf{g}}(\mathbf{z}) = \left[\frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\right]_{\mathbf{x}=\mathbf{\Phi}^{-1}(\mathbf{z})}, \qquad \overline{h}(\mathbf{z}) = h(\mathbf{\Phi}^{-1}(\mathbf{z})),$$

then

$$\begin{aligned} L_{\overline{\mathbf{f}}}\overline{h}(\mathbf{z}) = \frac{\partial \overline{h}}{\partial \mathbf{z}}\overline{\mathbf{f}}(\mathbf{z}) &= \left[\frac{\partial h}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{\Phi}^{-1}(\mathbf{z})} \left[\frac{\partial \mathbf{\Phi}^{-1}}{\partial \mathbf{z}}\right] \left[\frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})\right]_{\mathbf{x}=\mathbf{\Phi}^{-1}(\mathbf{z})} \\ &= \left[\frac{\partial h}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})\right]_{\mathbf{x}=\mathbf{\Phi}^{-1}(\mathbf{z})} = \left[L_{\mathbf{f}}h(\mathbf{x})\right]_{\mathbf{x}=\mathbf{\Phi}^{-1}(\mathbf{z})} \end{aligned}$$

then iterated calculations lead to

$$L_{\overline{\mathbf{f}}^k}\overline{h}(\mathbf{z}) = \left[L_{\mathbf{f}^k}h(\mathbf{x})\right]_{\mathbf{x}=\mathbf{\Phi}^{-1}(\mathbf{z})}$$

and

$$L_{\overline{\mathbf{g}}}L_{\overline{\mathbf{f}}^k}\overline{h}(\mathbf{z}) = \left[L_{\mathbf{g}}L_{\mathbf{f}^k}h(\mathbf{x})\right]_{\mathbf{x}=\mathbf{\Phi}^{-1}(\mathbf{z})}$$

[1] got by rewriting $A + Bu = v$

**Lemma 4** (Invariance of $r$ under feedback):
By induction, it may be shown that

$$L_{(\mathbf{f}+\mathbf{g}\alpha)^k} h(\mathbf{x}) = L_{\mathbf{f}^k} h(\mathbf{x}), \qquad k = 0, 1, \ldots, r-1$$

from which it follows that

$$L_{\mathbf{g}\beta} L_{(\mathbf{f}+\mathbf{g}\alpha)^k} h(\mathbf{x}) = 0, \qquad k = 0, 1, \ldots, r-2$$

and provided that $\beta(\mathbf{x}_e) \neq 0$

$$L_{\mathbf{g}\beta} L_{(\mathbf{f}+\mathbf{g}\alpha)^{r-1}} h(\mathbf{x}_e) \neq 0$$

To show the necessity of $r = n$, start by considering a linear system in *Brunovsky* canonical form [2]:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

If the nonlinear system is feedback linearisable then it can be transformed to an CC linear system which in turn can be transformed into *Brunovsky* canonical form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \quad \underset{u=\alpha+\beta v}{\overset{\mathbf{z}=\mathbf{\Phi}(\mathbf{x})}{\leftrightarrow}} \quad \dot{\mathbf{z}} = A\mathbf{z} + bv \quad \underset{v=k\mathbf{z}+w}{\overset{\mathbf{z}=\tilde{T}\tilde{\mathbf{z}}}{\leftrightarrow}} \quad \dot{\tilde{\mathbf{z}}} = \tilde{A}\tilde{\mathbf{z}} + \tilde{b}w$$

With this *Brunovsky* form system, associate the "output" $y = (1, 0, \ldots, 0)\tilde{\mathbf{z}}$. It is straightforward to show that $r = n$. The invariance of $r$ under transformation and feedback means that there is an appropriate output $y = \tilde{h}(\mathbf{x})$ associated with the nonlinear system.

The problem of finding $\tilde{h}(\mathbf{x})$ such that $r = n$ comes down to finding a solution to the set of equations/inequation:

$$\begin{aligned} L_{\mathbf{g}} L_{\mathbf{f}^k} \tilde{h}(\mathbf{x}) &= 0, \qquad k = 0, 1, \ldots, n-2 \\ L_{\mathbf{g}} L_{\mathbf{f}^{n-1}} \tilde{h}(\mathbf{x}_e) &\neq 0 \end{aligned}$$

[2] This is a particular instance of ccf

This in turn is equivalent to the set of 1st order equations/inequation:

$$L_{ad_{\mathbf{f}^k}g}\tilde{h}(\mathbf{x}) = 0, \qquad k = 0, 1, \dots, n-2 \tag{15}$$

$$L_{ad_{\mathbf{f}^{n-1}}g}\tilde{h}(\mathbf{x}_e) \neq 0 \tag{16}$$

Necessary and sufficient conditions for the existence of a function $\tilde{h}(\mathbf{x})$ satisfying these relations ( (15) and (16) ) is a consequence of *Frobenius*'s theorem and are given by

(i)
$$F_n = \{\mathbf{g}(\mathbf{x}), ad_{\mathbf{f}}\mathbf{g}(\mathbf{x}), \dots, ad_{\mathbf{f}^{n-2}}\mathbf{g}(\mathbf{x}), ad_{\mathbf{f}^{n-1}}\mathbf{g}(\mathbf{x})\}$$
is linearly independent in a neighbourhood of $\mathbf{x} = \mathbf{x}_e$. This corresponds to complete controllability in linear systems.

(ii)
$$F_{n-1} = \{\mathbf{g}(\mathbf{x}), ad_{\mathbf{f}}\mathbf{g}(\mathbf{x}), \dots, ad_{\mathbf{f}^{n-2}}\mathbf{g}(\mathbf{x})\}$$
is involutive in a neighbourhood of $\mathbf{x} = \mathbf{x}_e$. This is the condition that guarantees the solution to the set of PDEs (Eq (15)). It automatically holds for linear systems.

If conditions (i) and (ii) hold, solving the relations (15) and (16) gives

$$\tilde{h}(\mathbf{x}) = y = \mathbf{\Phi}_1(\mathbf{x}) = z_1,$$

the first component of the state transformation. The other components are given by (see normal form Eq( 7))

$$z_{i+1} = L_{\mathbf{f}}z_i, \qquad i = 1, 2, \dots, n-1$$

**Example 3** Consider the 2nd-order affine system (the same state equation as Ex.1 & Ex.2 but with no output.):

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} x_2 + 4x_1^2x_2 \\ -x_1 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} u$$

We calculate

$$ad_{\mathbf{f}}\mathbf{g} = [\mathbf{f}, \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x}) = \mathbf{0} - \begin{pmatrix} 8x_1x_2 & 1 + 4x_1^2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 - 4x_1^2 \\ 0 \end{pmatrix}$$

Hence

$$F_2 = \{\mathbf{g},\ ad_{\mathbf{f}}\mathbf{g}\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} -1-4x_1^2 \\ 0 \end{pmatrix} \right\}$$

$F_2$ is linearly independent by inspection and $F_1$ is involutive [3]. So, ...
Solve

$$L_{\mathbf{g}} z_1 = 0, \qquad L_{ad_{\mathbf{f}}\mathbf{g}} z_1 \neq 0$$

i.e.

$$L_{\mathbf{g}} z_1 = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial z_1}{\partial x_2} = 0 \qquad (17)$$

$$L_{ad_{\mathbf{f}}\mathbf{g}} z_1 = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \end{pmatrix} \begin{pmatrix} -1-4x_1^2 \\ 0 \end{pmatrix} = \left( \frac{\partial z_1}{\partial x_1} \right) (-1-4x_1^2) \neq 0 \qquad (18)$$

To satisfy InEQ (18), choose

$$\frac{\partial z_1}{\partial x_1} = 1 \qquad (19)$$

Then a solution of Eq (17) and Eq (19) is

$$z_1 = x_1. \qquad (20)$$

The rest of the solution then reduces to Example 1.
What happens if conditions (i) and (ii) are not satisfied? Then it is still possible to obtain by means of state feedback a partially linear system. As a matter of fact since $r < n$ using the linearising control $u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v$ the system becomes

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= v \\ \dot{\eta}_{r+1} &= q_{r+1}(\mathbf{z})) \\ \dot{\eta}_{r+2} &= q_{r+2}(\mathbf{z})) \\ &\vdots \\ \dot{\eta}_n &= q_n(\mathbf{z})) \\ y &= \xi_1 \end{aligned} \qquad (21)$$

[3] Every singleton set of vector fields is involutive

where $q_i(.) = D_i(\mathbf{\Phi}^{-1}(.)),\ i = r+1, \ldots, n$ (see Eq (11)).

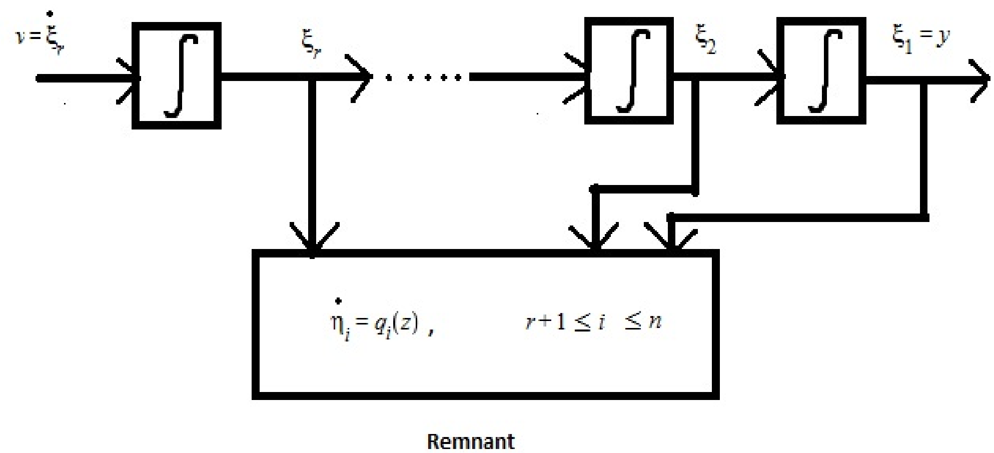


Figure 1: Normal Form