Optimal Control

1 The *Hamilton-Jacobi-Bellman* equation for Continuous-Time Systems

For the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

choose $\mathbf{u}(t), t_0 \leq t < t_f$ to minimise the performance index

$$J = \mathcal{T}(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u}, t)\, dt \tag{2}$$

where $\mathcal{T}(\mathbf{x}(t_f), t_f)$ is the terminal penalty and $L(\mathbf{x}, \mathbf{u}, t)$ is the lifetime penalty.
We embed this problem into a more general class of problems:
Define

(i)

$$J\left(\mathbf{x}(t), t, \mathbf{u}[t, t_f)\right) \triangleq \mathcal{T}(\mathbf{x}(t_f), t_f) + \int_{t}^{t_f} L(\mathbf{x}, \mathbf{u}, \tau)\, d\tau \tag{3}$$

where $\mathbf{u}[t, t_f) = \{\mathbf{u}(\tau) : t \leq \tau < t_f\}$ and

(ii)

$$V(\mathbf{x}(t), t) \triangleq \min_{\mathbf{u}[t, t_f)} \{J\left(\mathbf{x}(t), t, \mathbf{u}[t, t_f)\right)\} \tag{4}$$

$V(\mathbf{x}, t)$ is the optimal cost associated with starting in state $\mathbf{x}$ at time t.
By highlighting the interval $[t, t + \delta t)$ we can write

$$\begin{aligned} J\left(\mathbf{x}(t), t, \mathbf{u}[t, t_f)\right) &= \mathcal{T}(\mathbf{x}(t_f), t_f) + \int_{t}^{t+\delta t} L(\mathbf{x}, \mathbf{u}, \tau)\, d\tau + \int_{t+\delta t}^{t_f} L(\mathbf{x}, \mathbf{u}, \tau)\, d\tau \\ &= \int_{t}^{t+\delta t} L(\mathbf{x}, \mathbf{u}, \tau)\, d\tau + J\left(\mathbf{x}(t + \delta t), t + \delta t, \mathbf{u}[t + \delta t, t_f)\right) \end{aligned}$$

Hence

$$V(\mathbf{x}(t), t) = \min_{\mathbf{u}[t, t_f)} \{\int_{t}^{t+\delta t} L(\mathbf{x}, \mathbf{u}, \tau)\, d\tau + J\left(\mathbf{x}(t + \delta t), t + \delta t, \mathbf{u}[t + \delta t, t_f)\right)\} \tag{5}$$

Invoking the *Principle of Optimality*[1], Eq (5) becomes

$$\begin{aligned} V(\mathbf{x}(t), t) &= \min_{\mathbf{u}[t, t+\delta t)} \{\int_{t}^{t+\delta t} L(\mathbf{x}, \mathbf{u}, \tau)\, d\tau + \min_{\mathbf{u}[t+\delta t, t_f)} \{J\left(\mathbf{x}(t + \delta t), t + \delta t, \mathbf{u}[t + \delta t, t_f)\right)\}\} \\ &= \min_{\mathbf{u}[t, t+\delta t)} \{\int_{t}^{t+\delta t} L(\mathbf{x}, \mathbf{u}, \tau)\, d\tau + V(\mathbf{x}(t + \delta t), t + \delta t)\} \end{aligned} \tag{6}$$

[1] "An optimal policy has the property that what ever the initial state and initial decision are the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision." (Bellman)

If $V(\mathbf{x}(t+\delta t), t+\delta t)$ is analytic in a neighbourhood of $(\mathbf{x}(t), t)$, then

$$V(\mathbf{x}(t+\delta t), t+\delta t) \approx V(\mathbf{x}(t), t) + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \dot{\mathbf{x}}\delta t + \frac{\partial V}{\partial t}\delta t + \text{h.o.t}$$

Substituting this into Eq (6), cancelling $V(\mathbf{x}(t), t)$ on both sides, dividing through by δt and letting $\delta t \to 0$ gives

$$0 = \min_{\mathbf{u}(t)}\{L(\mathbf{x}, \mathbf{u}, t) + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \dot{\mathbf{x}} + \frac{\partial V}{\partial t}\}$$

or

$$0 = \min_{\mathbf{u}(t)}\{L(\mathbf{x}, \mathbf{u}, t) + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(\mathbf{x}, \mathbf{u}) + \frac{\partial V}{\partial t}\} \tag{7}$$

This partial differential equation is called the *Hamilton-Jacobi-Bellman* (HJB) equation. From Eqs (2) and (3), it has the associated boundary condition

$$V(\mathbf{x}(t_f), t_f) = \mathcal{T}(\mathbf{x}(t_f), t_f)$$

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2 Systems with infinite horizon

One of the major applications of optimal control is to systems where $t_f \to \infty$. In many such applications, it is required to choose a stabilising control over the interval $[0, \infty)$, i.e. $\mathbf{x}(t) \to \mathbf{0}$. Thus it is usually convenient to have $\mathcal{T}(\mathbf{x}(t_f), t_f) = 0$. The control problem is then: for the system of Eq (1), choose $\mathbf{u}(t), t_0 \le t < \infty$ to minimise

$$J = \int_{t_0}^{\infty} L(\mathbf{x}, \mathbf{u})\, dt \tag{8}$$

(Note that L does not explicitly depend on time t).
As before we embed this problem into a more general class of problems. We define

(i)

$$J(\mathbf{x}(t), \mathbf{u}[t, \infty)) \triangleq \int_{t}^{\infty} L(\mathbf{x}, \mathbf{u})\, d\tau \tag{9}$$

where $\mathbf{u}[t, \infty) = \{\mathbf{u}(\tau) : t \le \tau < \infty\}$ and

(ii)

$$V(\mathbf{x}(t)) \triangleq \min_{\mathbf{u}[t,\infty)} \{J(\mathbf{x}(t), \mathbf{u}[t, \infty))\} \tag{10}$$

Again note that V only depends on the state $\mathbf{x}$ and not explicitly on t. Thus $\frac{\partial V}{\partial t} = 0$ and so the HBJ equation (see Eq (7)) becomes

$$0 = \min_{\mathbf{u}(t)}\{L(\mathbf{x}, \mathbf{u}) + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(\mathbf{x}, \mathbf{u})\} \tag{11}$$