1 Solving Linear DEs

It is straightforward to verify that the differential equation

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{u}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

where $\mathbf{x}$ is a n-vector, A a $n \times n$ constant matrix and $\mathbf{u}$ an externally applied input, has solution

$$\mathbf{x}(t) = e^{A(t-t_0)} \left(\mathbf{x}_0 + \int_{t_0}^{t} e^{-A(\tau - t_0)} \mathbf{u}(\tau)\, d\tau \right) \tag{2}$$

where $e^{At} = \sum_{j=0}^{\infty} A^j \frac{t^j}{j!}$.
For autonomous Dynamical Systems, the external input is zero and thus the solution of

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3}$$

is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = P^{-1} e^{Jt} P \mathbf{x}_0 = \sum_{i=1}^{m} e^{\lambda_i t} \mathbf{g}_i(t) \tag{4}$$

where J is the *Jordan* form of A (see Review of Linear Algebra Topics); m is the number of distinct eigenvalues (λ_i) and $\mathbf{g}_i$ is a polynomial vector in t of degree at most $n - 1$. If A is diagonalisable, then $\mathbf{g}_i$ is a constant vector.