Linearisation

It is frequently advantageous to approximate nonlinear models by linear ones. Consider the continuous-time state model with given initial state

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 &(1)\\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, t) &(2)\end{aligned}$$

Furthermore, suppose that the reference or nominal input $\mathbf{u}^*(t)$ defined on the interval , $t_0 \le t < t_f$ generates corresponding solutions of Eq(1) and Eq(2) denoted by $\mathbf{x}^*(t)$ and $\mathbf{y}^*(t)$ respectively, defined on the same interval i.e.

$$\begin{aligned} \dot{\mathbf{x}}^*(t) &= \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) \qquad \mathbf{x}^*(t_0) = \mathbf{x}_0 &(3)\\ \mathbf{y}^*(t) &= \mathbf{h}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) &(4)\end{aligned}$$

Let us perturb the nominal value of the input as follows:

$$\mathbf{u} = \mathbf{u}^* + \delta\mathbf{u}$$

where $\delta\mathbf{u}$ is “small”. The corresponding state trajectory and output trajectory are perturbed to

$$\mathbf{x} = \mathbf{x}^* + \delta\mathbf{x} \qquad \mathbf{y} = \mathbf{y}^* + \delta\mathbf{y}$$

Since these perturbed variables satisfy the state and output equations, we get

$$\begin{aligned} \dot{\mathbf{x}}^* + \dot{\delta\mathbf{x}} &= \mathbf{f}(\mathbf{x}^* + \delta\mathbf{x}, \mathbf{u}^* + \delta\mathbf{u}, t) \qquad \mathbf{x}^*(t_0) + \delta\mathbf{x}(t_0) = \mathbf{x}_0\\ \mathbf{y}^* + \delta\mathbf{y} &= \mathbf{h}(\mathbf{x}^* + \delta\mathbf{x}, \mathbf{u}^* + \delta\mathbf{u}, t)\end{aligned}$$

Expanding $\mathbf{f}$ and $\mathbf{h}$ in *Taylor* series about $(\mathbf{x}^*,\ \mathbf{u}^*)$ yields

$$\begin{aligned} \dot{\mathbf{x}}^* + \dot{\delta\mathbf{x}} &= \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*, t) + A\,\delta\mathbf{x} + B\,\delta\mathbf{u} + \mathrm{O}(\delta\mathbf{x}, \delta\mathbf{u}) \qquad \mathbf{x}^*(t_0) + \delta\mathbf{x}(t_0) = \mathbf{x}_0\\ \mathbf{y}^* + \delta\mathbf{y} &= \mathbf{h}(\mathbf{x}^*, \mathbf{u}^*, t) + C\,\delta\mathbf{x} + D\,\delta\mathbf{u} + \mathrm{O}(\delta\mathbf{x}, \delta\mathbf{u})\end{aligned}$$

where the “Jacobian” matrices ($M = [m_{ij}]$)are defined by

$$A(t) = \left[\frac{\partial f_i}{\partial x_j}(\mathbf{x}^*, \mathbf{u}^*)\right] \qquad B(t) = \left[\frac{\partial f_i}{\partial u_j}(\mathbf{x}^*, \mathbf{u}^*)\right]$$

$$C(t) = \left[\frac{\partial h_i}{\partial x_j}(\mathbf{x}^*, \mathbf{u}^*)\right] \qquad D(t) = \left[\frac{\partial h_i}{\partial u_j}(\mathbf{x}^*, \mathbf{u}^*)\right]$$

and $\mathrm{O}(\delta\mathbf{x}, \delta\mathbf{u})$ represents higher order terms in $\delta\mathbf{x}$ and $\delta\mathbf{u}$.

Using Eq(3) and Eq(4), this simplifies to

$$\begin{aligned} \dot{\delta\mathbf{x}} &= A\,\delta\mathbf{x} + B\,\delta\mathbf{u} + \mathrm{O}(\delta\mathbf{x}, \delta\mathbf{u}) \qquad \delta\mathbf{x}(t_0) = \mathbf{0}\\ \delta\mathbf{y} &= C\,\delta\mathbf{x} + D\,\delta\mathbf{u} + \mathrm{O}(\delta\mathbf{x}, \delta\mathbf{u})\end{aligned}$$

Furthermore, if $(\delta\mathbf{x}, \delta\mathbf{u})$ are “small enough” to neglect the higher order terms, then we get (at least in some neighbourhood of $(\mathbf{x}^*, \mathbf{u}^*)$) the linearised model

$$\begin{aligned} \dot{\delta\mathbf{x}} &= A\,\delta\mathbf{x} + B\,\delta\mathbf{u} \qquad \delta\mathbf{x}(t_0) = \mathbf{0} &(5)\\ \delta\mathbf{y} &= C\,\delta\mathbf{x} + D\,\delta\mathbf{u} &(6)\end{aligned}$$

Solving Eq(5) and Eq(6) for $\delta\mathbf{x}(t)$ and $\delta\mathbf{y}(t)$, we obtain(at least in some neighbourhood of $(\mathbf{x}^*, \mathbf{u}^*)$) that

$$\mathbf{x}(t) \approx \mathbf{x}^*(t) + \delta\mathbf{x}(t) \qquad \mathbf{y}(t) \approx \mathbf{y}^*(t) + \delta\mathbf{y}(t)$$

One of the more common applications of Linearisation is in dealing with the stability of equilibrium states. If a system is in equilibrium at $\mathbf{x}_E$ with input $\mathbf{u}_E$, then its state does not change; so the state and output equations become

$$\mathbf{0} = \mathbf{f}(\mathbf{x}_E, \mathbf{u}_E, t) \tag{7}$$

$$\mathbf{y}_E = \mathbf{h}(\mathbf{x}_E, \mathbf{u}_E, t) \tag{8}$$

Given $\mathbf{u}_E$, we can solve Eq(7) and Eq(8) to get $\mathbf{x}_E$ and $\mathbf{y}_E$. The linearised model (Eq(5) and Eq(6)) is then calculated with $\mathbf{u}^* \equiv \mathbf{u}_E$, $\mathbf{x}^* \equiv \mathbf{x}_E$ and $\mathbf{y}^* \equiv \mathbf{y}_E$.

Example: The nonlinear inverted pendulum model described in Elgerd: *Control Systems Theory* (McGraw Hill 1967), Chapter 2, is, after elimination of some variables

$$\left(I + mL^2 \sin^2\phi + \frac{m}{m+M} ML^2 \cos^2\phi\right)\ddot{\phi} + \left(\frac{m}{m+M} mL^2 \sin\phi\cos\phi\right)\dot{\phi}^2$$

$$= \quad mgL\sin\phi - \left(\frac{m}{m+M} L\cos\phi\right) u$$

An equivalent state model is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b(x_1)}{a(x_1)} x_2^2 + \frac{c(x_1)}{a(x_1)} - \frac{d(x_1)}{a(x_1)} u \\ y &= x_1 \end{aligned}$$

where $x_1 = \phi$, $x_2 = \dot{\phi}$ and

$$\begin{aligned} a(x_1) &= I + mL^2 \sin^2 x_1 + \frac{m}{m+M} ML^2 \cos^2 x_1 \\ b(x_1) &= \frac{m}{m+M} mL^2 \sin x_1 \cos x_1 \\ c(x_1) &= mgL \sin x_1 \\ d(x_1) &= \frac{m}{m+M} L \cos x_1 \end{aligned}$$

There is an equilibrium state corresponding to $\phi(t) \equiv 0$ and $u(t) \equiv 0$. Thus we set

$$u^*(t) \equiv 0 \qquad \mathbf{x}^*(t) \equiv 0$$

Then the linearised model (with $\delta\mathbf{x} = \mathbf{x}$, $\delta\mathbf{u} = u$) is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g(m+M)mL}{I(m+M) + mML^2} x_1 - \frac{mL}{I(m+M) + mML^2} u \\ y &= x_1 \end{aligned}$$

or in vector-matrix notation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -\beta \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \end{bmatrix} u$$

with $\alpha = [g(m+M)mL] \,/\, [I(m+M) + mML^2]$ and $\beta = mL / \,[I(m+M) + mML^2]$.

The Linearisation process is analogous for discrete-time models.