**Linear Quadratic (LQ) Control**

# 1 Discrete-Time Systems

To measure the performance of the linear system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k \tag{1}$$

over the time interval $0, 1, 2, \ldots, N$, the quadratic performance index $J$ may be used where

$$J = \mathbf{x}_N^T S \mathbf{x}_N + \sum_{i=0}^{N-1} \mathbf{x}_i^T Q \mathbf{x}_i + \mathbf{u}_i^T R \mathbf{u}_i \tag{2}$$

Usually $R = R^T$ is positive definite, and $Q = Q^T$ and $S = S^T$ are positive semi-definite, so the index $J$ has the effect of penalising excessive control effort and excessive state magnitude. (Alternatively, $J$ has the effect of driving the state towards $\mathbf{0}$ while at the same time using the least amount of control effort.)
Thus we are interested in the LQ control problem of choosing
$\mathbf{u}_k, k = 0, 1, \ldots, N-1$ to minimise $J$ of Eq (2), for the system of Eq (1).

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To motivate the solution of this problem, consider the evaluation of $J$ when $\mathbf{u}_k \equiv \mathbf{0}$. If we define

$$J_k = \mathbf{x}_N^T S \mathbf{x}_N + \sum_{i=k}^{N-1} \mathbf{x}_i^T Q \mathbf{x}_i \tag{3}$$

we see that we are evaluating $J = J_0$. From Eq (3), we see that $J_k$ satisfies the backward difference equation:

$$J_k = \mathbf{x}_k^T Q \mathbf{x}_k + J_{k+1} \qquad J_N = \mathbf{x}_N^T S \mathbf{x}_N$$

It is a straightforward induction to obtain

$$J = \mathbf{x}_0^T P_0 \mathbf{x}_0 \qquad \text{where } P_k = Q + A^T P_{k+1} A, \quad P_N = S \tag{4}$$

(We note that $P_k = P_k^T$).
In the situation where the system is LTI and $N \to \infty$, inspection of the performance index shows that we need the system $A$ matrix to be a *convergence* matrix , i.e. all its eigenvalues must have magnitude less than 1, to ensure that the infinite sum converges. Under this assumption, $P_k \to P$ where $P$ is the positive definite solution of

$$P = Q + A^T P A \qquad (\Leftrightarrow A^T P A - P = -Q \text{ Discrete } \textit{Lyapunov} \text{ Equation})$$

and $J = \mathbf{x}_0^T P \mathbf{x}_0$

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Returning to the optimal control problem, define

$$J_k = \mathbf{x}_N^T S \mathbf{x}_N + \sum_{i=k}^{N-1} \mathbf{x}_i^T Q \mathbf{x}_i + \mathbf{u}_i^T R \mathbf{u}_i$$

$$V(\mathbf{x}_k) = \min_{\mathbf{u}_k, \mathbf{u}_{k+1}, \cdots, \mathbf{u}_{N-1}} (J_k)$$

As above

$$J_k = \mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k + J_{k+1}$$

Hence

$$V(\mathbf{x}_k) = \min_{\mathbf{u}_k, \mathbf{u}_{k+1}, \cdots, \mathbf{u}_{N-1}} (\mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k + J_{k+1}) \tag{5}$$

Invoking the *Principle of Optimality*: "Any portion of an optimal trajectory is also an optimal trajectory", Eq (5) gives

$$\begin{aligned} V(\mathbf{x}_k) &= \min_{\mathbf{u}_k} (\mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k + \min_{\mathbf{u}_{k+1}, \cdots, \mathbf{u}_{N-1}} (J_{k+1})) \\ &= \min_{\mathbf{u}_k} (\mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k + V(\mathbf{x}_{k+1})) \end{aligned} \tag{6}$$

This equation is called the *Hamilton-Jacobi-Bellman* (HJB) equation of *Dynamic Programming*. From Eq(2), it has the associated boundary condition $V(\mathbf{x}_N) = \mathbf{x}_N^T S \mathbf{x}_N$.

For the system of Eq (1), the solution of this equation is of the form

$$V(\mathbf{x}_k) = \mathbf{x}_k^T P_k \mathbf{x}_k \tag{7}$$

where $P_k = P_k^T$ is positive semi-definite. This may be shown using (a backwards) induction. With $P_N = S$, it obviously holds at $k = N$. Assume it holds at $k+1$. The right hand side of the HJB equation is then

$$\min_{\mathbf{u}_k} (F)$$

where

$$\begin{aligned} F &= \mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k + \mathbf{x}_{k+1}^T P_{k+1} \mathbf{x}_{k+1} \\ &= \mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k + (A\mathbf{x}_k + B\mathbf{u}_k)^T P_{k+1} (A\mathbf{x}_k + B\mathbf{u}_k) \end{aligned}$$

Since $\mathbf{u}_k$ is unconstrained, we minimise this by setting

$$\begin{aligned} \frac{\partial F}{\partial \mathbf{u}_k} &= \mathbf{0} \\ \Rightarrow (R + B^T P_{k+1} B)\mathbf{u}_k + B^T P_{k+1} A \mathbf{x}_k &= \mathbf{0} \end{aligned}$$

Given that $R$ is positive definite and $P_{k+1}$ positive semi-definite, then $R + B^T P_{k+1} B$ is positive definite (thus invertible) which yields that $\frac{\partial^2 F}{\partial \mathbf{u}_k^2} = R + B^T P_{k+1} B$ is positive definite implying that

$$\mathbf{u}_k = -(R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A \mathbf{x}_k \tag{8}$$

minimises $F$. Substituting this linear feedback law into the expression for $F$ yields

$$F_{min} = \mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{x}_k^T P_{k+1} A \mathbf{x}_k - \mathbf{x}_k^T A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A \mathbf{x}_k$$

Therefore by selecting $P_k$ according to ( the backwards difference equation DMRE: Discrete Matrix *Riccati* Equation)

$$P_k = Q + A^T P_{k+1} A - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A, \qquad P_N = S \tag{9}$$

we complete the induction. ( Why is $P_k$ positive semi-definite ?)

In summary, Eq(9) is solved backwards in time to generate $P_k$, then the feedback control is generated by Eq (8).

In the situation where the system of Eq (1) is LTI , $S = 0$ and $N \to \infty$, then under certain circumstances [1] $P_k \to P$ where $P$ is the positive definite solution of (the DARE: Discrete Algebraic *Riccati* Equation)

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA \tag{10}$$

Then the feedback control law is also time invariant:

$$\mathbf{u}_k = -(R + B^T PB)^{-1} B^T PA\mathbf{x}_k \tag{11}$$

## 2 Continuous-Time Systems

The LQ control problem in this scenario is: For the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{12}$$

Choose $\mathbf{u}(t), t_0 \leq t < t_f$ to minimise

$$J = \mathbf{x}(t_f)^T S\mathbf{x}_(t_f) + \int_{t_0}^{t_f} \mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u}\, dt \tag{13}$$

The performance index has the same interpretation as for the discrete-time case.

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To motivate the solution of this problem, consider the evaluation of $J$ when $\mathbf{u}(t) \equiv \mathbf{0}$. If we define

$$J(\mathbf{x}(t), t) = \mathbf{x}(t_f)^T S\mathbf{x}(t_f) + \int_t^{t_f} \mathbf{x}(\tau)^T Q\mathbf{x}(\tau)\, d\tau \tag{14}$$

we see that we are evaluating $J = J(x_0, t_0)$. From Eq (14), $J(\mathbf{x}(t), t)$ satisfies the differential equation and terminal(boundary) condition:

$$\frac{dJ(\mathbf{x}, t)}{dt} = -\mathbf{x}^T Q\mathbf{x}, \qquad J(\mathbf{x}(t_f), t_f) = \mathbf{x}(t_f)^T S\mathbf{x}(t_f)$$

Assuming that $J(\mathbf{x}(t), t) = \mathbf{x}^T(t)P(t)\mathbf{x}(t)$ we obtain that

$$J = \mathbf{x}_0^T P(t_0)\mathbf{x}_0 \qquad \text{where } \dot{P} + A^T P + PA = -Q, \quad P(t_f) = S \tag{15}$$

(We note that $P = P^T$. To obtain $P(t)$, integrate the differential equation backwards in time.)
In the situation where the system is LTI and $t_f \to \infty$, inspection of the performance index shows that we need the system $A$ matrix to be a *stability* matrix , i.e. all its eigenvalues must have real part less than 0, to ensure that the infinite sum converges. Under this assumption, $P(t_0) \to P$ where $P$ is the positive definite solution of

$$A^T P + PA = -Q \quad (\textit{Lyapunov} \ \text{Equation})$$

[1] One set of such circumstances is if $(A, B)$ is stabilisable and $(A, C)$ is detectable, where $C$ is such that $C^T C = Q$.

and $J = \mathbf{x}_0^T P \mathbf{x}_0$.

Returning to the optimal control problem, define

$$J(\mathbf{x}(t), t, \mathbf{u}[t, t_f)) = \mathbf{x}(t_f)^T S \mathbf{x}(t_f) + \int_t^{t_f} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} \, d\tau$$

where $\mathbf{u}[t, t_f) = \{\mathbf{u}(\tau) : t \leq \tau < t_f\}$ and

$$V(\mathbf{x}(t), t) = \min_{\mathbf{u}[t,t_f)} (J(\mathbf{x}(t), t, \mathbf{u}[t, t_f)))$$

By considering the interval $[t, t + \delta t)$ we get

$$\begin{aligned} J(\mathbf{x}(t), t, \mathbf{u}[t, t_f)) &= \mathbf{x}(t_f)^T S \mathbf{x}(t_f) + \int_t^{t+\delta t} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} \, d\tau + \int_{t+\delta t}^{t_f} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} \, d\tau \\ &= \int_t^{t+\delta t} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} \, d\tau + J(\mathbf{x}(t + \delta t), t + \delta t, \mathbf{u}[t + \delta t, t_f)) \end{aligned}$$

Hence

$$V(\mathbf{x}(t), t) = \min_{\mathbf{u}[t,t_f)} (\int_t^{t+\delta t} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} \, d\tau + J(\mathbf{x}(t+\delta t), t+\delta t, \mathbf{u}[t+\delta t, t_f))) \quad (16)$$

Invoking the *Principle of Optimality*: Eq (16) gives

$$\begin{aligned} V(\mathbf{x}(t), t) &= \min_{\mathbf{u}[t,t+\delta t)} (\int_t^{t+\delta t} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} \, d\tau + \min_{\mathbf{u}[t+\delta t,t_f)} (J(\mathbf{x}(t + \delta t), t + \delta t, \mathbf{u}[t + \delta t, t_f))))) \\ &= \min_{\mathbf{u}[t,t+\delta t)} (\int_t^{t+\delta t} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} \, d\tau + V(\mathbf{x}(t + \delta t), t + \delta t)) \end{aligned} \quad (17)$$

If $V(\mathbf{x}(t + \delta t), t + \delta t)$ is analytic in a neighbourhood of $(\mathbf{x}(t), t)$, then

$$V(\mathbf{x}(t + \delta t), t + \delta t) \approx V(\mathbf{x}(t), t) + \frac{\partial V}{\partial \mathbf{x}} \dot{\mathbf{x}} \delta t + \frac{\partial V}{\partial t} \delta t + \text{h.o.t}$$

Substituting this into Eq (17), cancelling $V(\mathbf{x}(t), t)$ on both sides, dividing through by $\delta t$ and letting $\delta t \to 0$ gives

$$0 = \min_{\mathbf{u}(t)} (\mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} + \frac{\partial V}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial V}{\partial t}) \quad (18)$$

This partial differential equation is called the *Hamilton-Jacobi* (HJ) equation. From Eq(13), it has the associated boundary condition $V(\mathbf{x}(t_f), t_f) = \mathbf{x}(t_f)^T S \mathbf{x}(t_f)$.

For the system of Eq (12), we can find a solution of this equation of the form

$$V(\mathbf{x}(t), t) = \mathbf{x}(t)^T P(t) \mathbf{x}(t) \quad (19)$$

where $P = P^T$ is positive semi-definite. This may be shown as follows: The boundary condition is satisfied by $P(t_f) = S$. The right hand side of the HJ equation is then

$$\min_{\mathbf{u}(t)} (F)$$

where

$$\begin{aligned} F &= \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} + \dot{\mathbf{x}}^T P \mathbf{x} + \mathbf{x}^T P \dot{\mathbf{x}} + \mathbf{x}^T \dot{P} \mathbf{x} \\ &= \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} + (A\mathbf{x} + B\mathbf{u})^T P \mathbf{x} + \mathbf{x}^T P (A\mathbf{x} + B\mathbf{u}) + \mathbf{x}^T \dot{P} \mathbf{x} \end{aligned}$$

Since $\mathbf{u}$ is unconstrained, we minimise this by setting

$$\begin{aligned} \frac{\partial F}{\partial \mathbf{u}} &= \mathbf{0} \\ \Rightarrow R\mathbf{u} + B^T P\mathbf{x} &= \mathbf{0} \end{aligned}$$

Given that $R$ is positive definite (thus invertible), this that $\frac{\partial^2 F}{\partial \mathbf{u}^2} = R$ is positive definite implying that

$$\mathbf{u} = -R^{-1}B^T P(t)\mathbf{x} \tag{20}$$

minimises $F$. Substituting this linear feedback law into the expression for $F$ yields

$$F_{min} = \mathbf{x}^T Q\mathbf{x} - \mathbf{x}^T P(t)BR^{-1}B^T P(t)\mathbf{x} + \mathbf{x}^T A^T P(t)\mathbf{x} + \mathbf{x}^T P(t)A\mathbf{x} + \mathbf{x}^T \dot{P}(t)\mathbf{x}$$

Therefore by selecting $P(t)$ to satisfy the differential equation (MRE: Matrix *Riccati* Equation)

$$0 = Q - P(t)BR^{-1}B^T P(t) + A^T P(t) + P(t)A + \dot{P}(t), \qquad P(t_f) = S \tag{21}$$

we prove our assertion. ( Why is $P(t)$ positive semi-definite ?)

In summary, Eq(21) is solved backwards in time to generate $P(t)$, then the feedback control is generated by Eq (20).

In the situation where the system of Eq (12) is LTI , $S = 0$ and $t_f \to \infty$, then under certain circumstances (generally the same as for the discrete-time case), $P(t) \to P$ where $P$ is the positive definite solution of (the ARE: Algebraic *Riccati* Equation)

$$0 = Q - PBR^{-1}B^T P + A^T P + PA, \tag{22}$$

Then the feedback control law is also time invariant:

$$\mathbf{u} = -R^{-1}B^T P\mathbf{x} \tag{23}$$