LAPLACE TRANSFORMS

1. Basic transforms

In this course, Laplace Transforms will be introduced and their properties examined; a table of common transforms will be built up; and transforms will be used to solve some differential equations by transforming them into algebraic equations which can be easily solved.

Let $f(t)$ be defined for $t \geq 0$. Then we define the **Laplace Transform** of f as

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$$

We also define the inverse transform $\mathcal{L}^{-1}$ by

$$f = \mathcal{L}^{-1} F$$

Remark 1.1. $\mathcal{L}$ is an **operator** on the **function** f. Whereas f depends on the variable t, F is independent of t.

Example. $f(t) \equiv 1, \quad t \geq 0$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}\, dt \\ &= -\frac{1}{s}\left[e^{-st}\right]_0^\infty \\ &= \frac{1}{s} \end{aligned}$$

Thus we say that $\mathcal{L}(1) = \dfrac{1}{s}$

1.1. Exponentials.

Example $f(t) = e^{at}, \quad t \geq 0$, a

$$\begin{aligned} \mathcal{L}(e^{at}) &= \int_0^\infty e^{-st} e^{at}\, dt \\ &= \int_0^\infty e^{(a-s)t}\, dt \\ &= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty \\ &= \frac{1}{s-a} \quad \text{if } s > a. \end{aligned}$$

Linearity of Laplace Transform

Let $f(t)$ and $g(t)$ be defined for $t \geq 0$ with transforms $F(s) = \mathcal{L}(f)$ and $G(s) = \mathcal{L}(g)$ respectively, and let a and b be constants. Then

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

PROOF.

$$\begin{aligned}\mathcal{L}\{af(t)+bg(t)\} &= \int_0^\infty e^{-st}[af(t)+bg(t)]\,dt \\ &= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt \\ &= aF(s)+bG(s)\end{aligned}$$

□

Example. $\cosh t = \dfrac{e^{at}+e^{-at}}{2}$

$$\begin{aligned}\mathcal{L}(\cosh t) &= \frac{1}{2}\{\mathcal{L}e^{at}+\mathcal{L}e^{-at}\} \\ &= \frac{1}{2}\left\{\frac{1}{s-a}+\frac{1}{s+a}\right\}\end{aligned}$$

when $s-a>0$ and $s+a>0$. We summarise this as

$$\mathcal{L}(\cosh at) = \frac{s}{s^2-a^2} \quad s>a\geq 0$$

1.2. Powers of t.

$$\begin{aligned}\mathcal{L}(t^a) &= \int_0^\infty e^{-st}t^a\,dt \\ &= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^a\frac{dx}{s} \quad \text{setting } x=st \\ &= \frac{1}{s^{a+1}}\int_0^\infty e^{-x}x^a\,dx\end{aligned}$$

or

$$\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}$$

where $\Gamma(a)$ is the **Gamma function** defined by

$$\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}\,dx$$

for $a>0$. Integrating by parts:

$$\begin{aligned}\Gamma(a+1) &= \int_0^\infty e^{-x}x^a\,dx \\ &= -\left[x^a e^{-x}\right]_0^\infty + a\int_0^\infty e^{-x}x^{a-1}\,dx \\ &= a\Gamma(a)\end{aligned}$$

REMARK 1.2. $\Gamma(1)=\int_0^\infty e^{-x}\,dx = -\left[e^{-x}\right]_0^\infty = 1$

Let n be a positive integer. Then

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &= n(n-1)(n-2)\Gamma(n-2) \\ &\ \ \vdots \\ &= n(n-1)(n-2)\dots(3)(2)(1)\Gamma(1) \\ &= n!\end{aligned}$$

Thus for a positive integer n we have

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

We also have $\Gamma(1/2) = \sqrt{\pi}$.

Even though the Gamma function is so far only defined for positive values of a, we can extend the definition to (some) negative values of a using the property $\Gamma(a+1) = a\Gamma(a)$. For example,

$$\Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$$

and

$$\Gamma(-3/2) = \frac{\Gamma(-1/2)}{-3/2} = 4\sqrt{\pi}/3$$

However, $\Gamma(1) = 0\Gamma(0)$ and so $\Gamma(0) = 1/0$ which is unbounded. Similarly $\Gamma(n)$ is unbounded for all negative integers n.

1.3. Trigonometric functions. We already have $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$ and so

$$\begin{aligned}\mathcal{L}(e^{i\omega t}) &= \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2} \\ &= \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}\end{aligned}$$

and since $e^{i\omega t} = \cos\omega t + i\sin\omega t$ we have

$$\mathcal{L}(\cos\omega t + i\sin\omega t) = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

and so, equating real and imaginary parts, we have

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}$$

and

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$$

1.4. Step function.

$$f(t) = \begin{cases} k, & 0 \le t < c; \\ 0, & c \le t \end{cases}$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^c ke^{-st}\, dt = -\frac{k}{s}\left[e^{-st}\right]_0^c \\ &= \frac{k}{s}[1 - e^{-sc}]\end{aligned}$$

1.5. Laplace transforms of derivatives. Let $f(t)$ be differentiable for $t \geq 0$ with transform $\mathcal{L}\{f(t)\}$. Then $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$ Integrating by parts, with

$$u = e^{-st} \quad dv = f'(t)\, dt$$
$$du = -s\, e^{-st}\, dt \qquad v = f(t)$$

$$\begin{aligned} \Rightarrow \mathcal{L}f'(t) &= \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt \\ &= s\mathcal{L}f(t) - f(0) \end{aligned}$$

Similarly, if f is twice differentiable, then

$$\begin{aligned} \mathcal{L}\{f''(t)\} &= s\mathcal{L}f'(t) - f'(0) \\ &= s\left[s\mathcal{L}f(t) - f(0)\right] - f'(0) \\ &= s^2\mathcal{L}f(t) - sf(0) - f'(0) \end{aligned}$$

and in general

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

To verify this, we can see for example that if $f(t) = t^3$, then

$$f(0) = f'(0) = f''(0) = 0 \qquad f'''(t) = 6$$

Then

$$\begin{aligned} \mathcal{L}\{f'''(t)\} &= \mathcal{L}(6) = 6/s \\ &= s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0) = s^3\mathcal{L}(t^3) \end{aligned}$$

and so

$$\mathcal{L}(t^3) = 6/s^4 = 3!/s^4$$

as already seen.

Example (Initial Value Problem)

$$y'' + 5y' + 6y = 0 \quad y(0) = 2, \quad y'(0) = 3$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ Then

$$\mathcal{L}\{y'(t)\} = sY - y(0) = sY - 2$$
$$\mathcal{L}\{y''(t)\} = s^2Y - sy(0) - y'(0) = s^2Y - 2s - 3$$

and transforming the equation gives

$$s^2Y - 2s - 3 + 5(sY - 2) + 6Y = 0$$

or

$$(s^2 + 5s + 6)Y = 2s + 13$$

which is called the *subsidiary equation.* This has solution

$$Y = \frac{2s + 13}{(s+3)(s+2)}$$

which now must be inverted to obtain the solution $y(t)$.

Using partial fractions

$$Y = \frac{9}{s+2} - \frac{7}{s+3}$$

and thus

$$y = \mathcal{L}^{-1}(Y) = 9\mathcal{L}^{-1}\{\frac{1}{s+2}\} - 7\mathcal{L}^{-1}\{\frac{1}{s+3}\} = 9e^{-2t} - 7e^{-3t}$$

1.6. The shifting theorems. To solve more complicated problems we need the following.

THEOREM 1.3. *(**First Shifting Theorem.**) If $\mathcal{L}\{f(t)\} = F(s)$ for $s > \gamma$; then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ for $s > \gamma + a$*

PROOF. $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$\begin{aligned} F(s-a) &= \int_0^\infty e^{-(s-a)t} f(t)\, dt \\ &= \int_0^\infty e^{-st} e^{at} f(t)\, dt \\ &= \mathcal{L}\{e^{at} f(t)\} \end{aligned}$$

□

Example. $\mathcal{L}(1) = 1/s = F(s)$ for $s > 0$. Thus $\mathcal{L}(e^{at}) = F(s-a) = 1/(s-a)$ for $s > a$ as seen already.

Example.

$$y'' + 2y' + 5y = 0, \qquad y(0) = 2, \quad y'(0) = -4$$

Transforming ⇒

$$\begin{aligned} & s^2 Y - 2s + 4 + 2(sY - 2) + 5Y = 0 \\ \Rightarrow\ & (s^2 + 2s + 5)Y = 2s \\ \Rightarrow\ & Y = \frac{2s}{(s+1)^2 + 2^2} \\ & \quad = 2\frac{s+1}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 2^2} \end{aligned}$$

From before, we know that

$$\mathcal{L}^{-1}\{\frac{s}{s^2 + 2^2}\} = \cos 2t$$

and

$$\mathcal{L}^{-1}\{\frac{2}{s^2 + 2^2}\} = \sin 2t$$

and so, using the first shifting theorem this implies

$$\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2 + 2^2}\} = e^{-t} \cos 2t$$

and

$$\mathcal{L}^{-1}\{\frac{2}{(s+1)^2 + 2^2}\} = e^{-t} \sin 2t$$

and so $y = \mathcal{L}^{-1}(Y) = e^{-t}[2\cos 2t - \sin 2t]$

The analogue of the first shifting theorem for inverse transforms is the second shifting theorem. This makes use of the *unit step function*:

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

This is also sometimes written as $u(t-a)$ or as the *Heaviside function* $H_a(t)$ or $H(t-a)$.

THEOREM 1.4. *(**Second shifting theorem**) If $\mathcal{L}^{-1}(F(s)) = f(t)$, then*

$$\begin{aligned} \mathcal{L}^{-1}(e^{-as}F(s)) = \tilde{f}(t) &= \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases} \\ &= f(t-a)u_a(t) \end{aligned}$$

PROOF.

$$\begin{aligned} e^{-as}F(s) &= e^{-as}\int_0^\infty e^{-sz}f(z)\,dz \\ &= \int_0^\infty e^{-s(a+z)}f(z)\,dz \\ \underline{t = a+z} \quad &= \int_a^\infty e^{-st}f(t-a)\,dt \\ &= \int_0^\infty e^{-st}f(t-a)u_a(t)\,dt \\ &= \mathcal{L}[f(t-a)u_a(t)] \end{aligned}$$

□

Example. $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s^2+1}\right]$ Since $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t$, the second shifting theorem implies that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+1}\right] &= \sin(t-3)u_3(t) \\ &= \begin{cases} 0, & t < 3 \\ \sin(t-3), & t \geq 3 \end{cases} \end{aligned}$$

REMARK 1.5.

$$\begin{aligned} \mathcal{L}\{u_a(t)\} &= \int_0^\infty e^{-st}u_a(t)\,dt \\ &= \int_a^\infty e^{-st}\,dt \\ &= \left[-\frac{1}{s}e^{-st}\right]_a^\infty \\ &= \frac{e^{-as}}{s} \end{aligned}$$

REMARK 1.6. The "delta function" $\delta_a(t)$ (also written $\delta(t-a)$) which is infinite at the point $t = a$ and zero elsewhere is the first derivative of the unit step function $u_a(t)$. Thus

$$\mathcal{L}\{\delta_a(t)\} = \mathcal{L}\{u_a'(t)\} = s\mathcal{L}\{u_a(t)\} - u_a(0) = e^{-as}$$

1.7. Transforms of integrals.

THEOREM 1.7. *If f has transform $\mathcal{L}(f) = F(s)$ then*

$$\mathcal{L}\left[\int_0^t f\right] = \frac{F(s)}{s}$$

PROOF. Let $g(t) = \int_0^t f$. Then $g'(t) = f(t)$ and $g(0) = 0$ and so

$$\begin{aligned}\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} &= s\mathcal{L}\{g(t)\} - g(0)\\ &= s\mathcal{L}\{g(t)\}\end{aligned}$$

and so

$$\mathcal{L}\{g(t)\} = \mathcal{L}\left[\int_0^t f\right] = \frac{1}{s}\mathcal{L}\{f(t)\}$$

□

Example. Invert $\frac{s-1}{s^2(s+1)}$.

$$\mathcal{L}^{-1}\left[\frac{s-1}{s^2(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right]$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] &= e^{-t}\\ \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] &= \int_0^t e^{-z}\,dz = 1 - e^{-t}\\ \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right] &= \int_0^t (1 - e^{-z})\,dz\\ &= \left[z + e^{-z}\right]_0^t = t + e^{-t} - 1\\ \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{s-1}{s^2(s+1)}\right] &= 2(1 - e^{-t}) - t\end{aligned}$$

We can use these results to solve more complicated initial value problems, which could not be solved without the use of transforms:

Example. Response of an undamped system to a single square wave.

$$\begin{aligned}&y'' + 2y = r(t)\\ &y(0) = y'(0) = 0\\ &r(t) = \begin{cases}1, & t < 1\\ 0, & t > 1\end{cases}\end{aligned}$$

Thus $r(t) = 1 - u_1(t)$ and $\mathcal{L}\{r(t)\} = \frac{1 - e^{-s}}{s}$. Setting $Y = \mathcal{L}(y)$ and transforming the equation we obtain

$$s^2Y + 2Y = \frac{1}{s}(1 - e^{-s})$$

and so

$$Y = \frac{1}{s(s^2+2)}[1 - e^{-s}]$$

To find y we first invert

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2}\right] = \frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

and so

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s(s^2+2)}\right] &= \frac{1}{\sqrt{2}}\int_0^t \sin\sqrt{2}z\,dz \\
&= -\frac{1}{2}\left[\cos\sqrt{2}z\right]_0^t \\
&= \frac{1}{2}\left(1 - \cos\sqrt{2}t\right)
\end{aligned}$$

We set $f(t) = \frac{1}{2}\left(1 - \cos\sqrt{2}t\right)$ and therefore, using the second shifting theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{e^{-s}}{s(s^2+2)}\right] &= u_1(t)f(t-1) \\
&= \frac{1}{2}\left[1 - \cos\sqrt{2}(t-1)\right]u_1(t)
\end{aligned}$$

Thus

$$y = \frac{1}{2}\left[1 - \cos\sqrt{2}t\right] - \frac{1}{2}\left[1 - \cos\sqrt{2}(t-1)\right]u_1(t)$$

For $t < 1$, $u_1(t) = 0$ and so

$$y = \frac{1}{2}\left[1 - \cos\sqrt{2}t\right]$$

whereas, for $t \geq 1$, $u_1(t) = 1$ and so

$$y = \frac{1}{2}\left[\cos\sqrt{2}(t-1) - \cos\sqrt{2}t\right]$$

Remark 1.8. Instead of using integration, we could have used partial fractions:

$$\frac{1}{s(s^2+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$$

or

$$1 = A(s^2+2) + (Bs+C)s$$

For $s = 0$, $1 = 2a$ and so $A = 1/2$. Equating coefficients of powers of s gives

$$\begin{aligned}
s^2 &: \quad 0 = A + B \quad \Rightarrow \quad B = -1/2 \\
s &: \quad 0 = C
\end{aligned}$$

Thus

$$\frac{1}{s(s^2+2)} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2}\right]$$

and so

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+2)}\right] = \frac{1}{2}\left[1 - \cos\sqrt{2}t\right]$$

as before.

Example. RC circuit

$$Ri(t) + \frac{1}{C}\int_0^t i = v(t)$$

$$\text{where } v(t) = \begin{cases} 0, & t < a \\ V_0, & a < t < b \\ 0, & t > b \end{cases}$$

That is, $v(t) = V_0[u_a(t) - u_b(t)]$ If we set $I(s) = \mathcal{L}[i(t)]$ then the transformed equation is

$$RI + \frac{I}{sC} = \frac{V_0}{s}\left[e^{-as} - e^{-bs}\right]$$

and so

$$I = F(s)(e^{-as} - e^{-bs})$$

where $F(s) = \dfrac{V_0}{Rs + 1/C}$

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{V_0}{Rs + 1/C}\right] &= \mathcal{L}^{-1}\left[\frac{V_0/R}{s + 1/RC}\right] \\ &= \frac{V_0}{R}\mathcal{L}^{-1}\left[\frac{1}{s + 1/RC}\right] \\ &= \frac{V_0}{R}e^{-t/RC}\end{aligned}$$

Thus

$$\begin{aligned}i(t) &= \mathcal{L}^{-1}[I(s)] = \mathcal{L}^{-1}[e^{-as}F(s)] - \mathcal{L}^{-1}[e^{-bs}F(s)] \\ &= \frac{V_0}{R}\left\{e^{-(t-a)/RC}u_a(t) - e^{-(t-b)/RC}u_b(t)\right\}\end{aligned}$$

So, for $t < a$, $u_a = u_b = 0$ and so $i(t) = 0$. For $a < t < b$ $u_a = 1$ but $u_b = 0$ still and so

$$i(t) = \left[\frac{V_0}{R}e^{a/RC}\right]e^{-t/RC}$$

and finally, for $t > b$, $u_a = u_b = 1$ and thus

$$\frac{V_0}{R}\left[e^{a/RC} - e^{b/RC}\right]e^{-t/RC}$$

1.8. Periodic functions. A function $f(t)$ has period T if $f(t + T) = f(t)$ for all t. For example, $\sin t$ and $\cos t$ have period 2π. We already know the transforms of these functions; but to transform more general (including discontinuous) periodic functions we do the following:

$$\begin{aligned}\mathcal{L}[f(t)] &= \int_0^\infty e^{-st}f(t)\,dt \\ &= \int_0^T e^{-st}f(t)\,dt + \int_T^{2T} e^{-st}f(t)\,dt + \int_{2T}^{3T} e^{-st}f(t)\,dt + \ldots \\ &\ldots + \int_{(n-1)T}^{nT} e^{-st}f(t)\,dt + \ldots\end{aligned}$$

Setting $t = u + T$ in the first integral, $t = u + 2T$ in the second, and so on, with $t = u + (n-1)T$ in the nth, we obtain

$$\mathcal{L}[f(t)] = \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-s(u+T)} f(u)\,du + \int_0^T e^{-s(u+2T)} f(u)\,du + \ldots + \int_0^T e^{-s(u+(n-1)T)} f(u)\,du + \ldots$$
$$= \left[1 + e^{-sT} + e^{-2sT} + \ldots\right] \int_0^T e^{-su} f(u)\,du$$

(using the periodicity of f). Noting that the term in square brackets is a geometric series, we can sum it to obtain finally

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Example. Periodic square wave

$$f(t) = \begin{cases} k, & 2na < t < (2n+1)a \\ -k, & (2n+1)a < t < 2(n+1)a \end{cases}$$

for $n = 0, 1, 2, \ldots$, which has period $2a$.

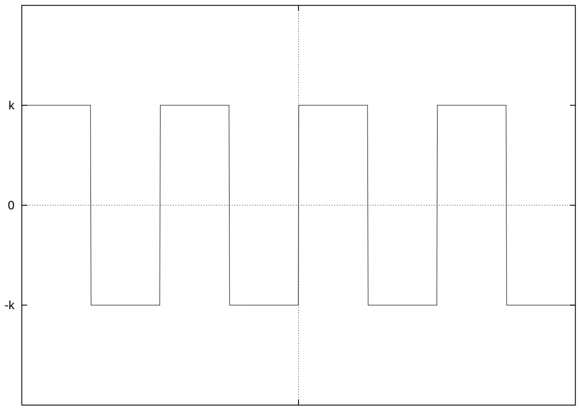

FIGURE 1. $f(t)$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}} \left[\int_0^a ke^{-st}\,dt - \int_a^{2a} ke^{-st}\,dt\right]$$
$$= \frac{k[1 - e^{-as} - e^{-as} + e^{-2as}]}{s(1-e^{-2as})}$$
$$= \frac{k}{s} \frac{[1-e^{-as}]^2}{[1-e^{-as}][1+e^{-as}]}$$
$$= \frac{k}{s} \frac{1-e^{-as}}{1+e^{-as}} \quad (= \frac{k}{s} \tanh \frac{as}{2})$$

1.9. Differentiation of transforms. If $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ then

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\,dt = -\mathcal{L}[tf(t)]$$

Hence, $\mathcal{L}[tf(t)] = -F'(s)$

Example. Find the inverse transform of $\frac{s}{(s^2+a^2)^2}$

Noting that $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, we have

$$\mathcal{L}[t \sin at] = \frac{2as}{(s^2+a^2)^2}$$

and so

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

Example.

$$F(s) = \ln\frac{s+a}{s-a} = \ln(s+a) - \ln(s-a)$$

Differentiating:

$$-\frac{d}{ds}\left[\ln\frac{s+a}{s-a}\right] = \frac{1}{s-a} - \frac{1}{s+a} = \frac{2a}{s^2-a^2}$$
$$= \mathcal{L}[e^{at} - e^{-at}] = 2\mathcal{L}[\sinh at]$$

$$\Rightarrow \quad \mathcal{L}^{-1}[F(s)] = \frac{e^{at}-e^{-at}}{t} = 2\frac{\sinh at}{t}$$

1.10. Integration of transforms. From this rule for differentiation of transforms, we can obtain the following rule for integration:

THEOREM 1.9. *If* $F(s) = \mathcal{L}[f(t)]$ *and* $\lim_{t\to 0_+}\frac{f(t)}{t}$ *exists, then*

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F$$

PROOF.

$$\int_s^\infty F(u)\,du = \int_s^\infty \left[\int_0^\infty e^{-ut}f(t)\,dt\right] du$$
$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-ut}\,du\right] dt$$
$$= \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

□

Example. We know that $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$ and $\lim_{t\to 0}\dfrac{\sin at}{t} = a$. Thus

$$\mathcal{L}\{\frac{sinat}{t}\} = \int_s^\infty \frac{a}{u^2+a^2}\,du$$
$$= -\left[\cot^{-1}\frac{u}{a}\right]_s^\infty = \cot^{-1}\frac{s}{a}$$

At this stage we can summarise all the transforms which have been established in the following table.

TABLE 1. Table of Laplace Transforms

$f(t)$	$F(s)$
0	0
1	$\frac{1}{s}$
k	$\frac{k}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
t^α	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$u_a(t)f(t-a)$	$e^{-as}F(s)$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{(n-1)}(0)$
$\int_0^t f(u)\,du$	$\frac{F(s)}{s}$
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

1.11. Systems of linear differential equations. These present no special difficulties for Laplace transforms. The transformed system of linear algebraic equations is solved and then inverted.

Example. A circuit is described by the system of two equations:

$$L_1i_1' + i_1R_1 = Mi_2' + v(t)$$
$$L_2i_2' + i_2R_2 = Mi_1'$$

If $L_1 = L_2 = 2$, $M = 1$ and $R_1 = R_2 = 3$, then this becomes

$$2i_1' + 3i_1 = i_2 + v(t)$$
$$2i_2' + 3i_2 = i_1'$$

Assuming $i_1(0) = i_2(0) = 0$, the transformed equation is then

$$2sI_1 + 3I_1 = sI_2 + V(s)$$
$$2sI_2 + 3I_2 = sI_1$$

or

$$(2s+3)I_1 - sI_2 \quad = V(s) \quad (1)$$
$$-sI_1 + (2s + 3I_2) \quad = 0 \quad (2)$$

Equation 2 $\Rightarrow$

$$\left[\frac{(2s+3)^2}{s} - s\right] I_2 = V(s)$$

and so

$$I_2 = \frac{sV(s)}{(2s+3)^2 - s^2} = \frac{sV(s)}{3(s+1)(s+3)}$$

Consider now the case where $v(t) = E \sin t$ for some constant E.
Then $V(s) = \dfrac{E}{s^2+1}$ and so

$$I_2 = \frac{E}{3}\left[\frac{s}{(s+1)(s+3)(s^2+1)}\right]$$

Using partial fractions:

$$\frac{s}{(s+1)(s+3)(s^2+1)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

or

$$s = A(s+3)(s^2+1) + B(s+1)(s^2+1) + (Cs+D)(s+1)(s+3)$$

Then $s = -1 \Rightarrow -1 = 4A$ and so $A = -1/4$
$s = -3 \Rightarrow -3 = -20B \Rightarrow B = 3/20$
Equating the coefficients of s^3 we have: $0 = A + B + C$ and so $C = -A - B = (5-3)/20 = 1/10$
and finally setting $s = 0$ we obtain $0 = 3A + B + 3D$ or $D = -1/3(3A+B) = -\dfrac{1}{3}\left(\dfrac{-15+3}{20}\right) = 1/5$
Thus

$$I_2 = \frac{E}{60}\left[-\frac{5}{s+1} + \frac{3}{s+3} + \frac{2s}{s^2+1} + \frac{4}{s^2+1}\right]$$

and inverting we obtain

$$i_2 = \frac{E}{60}\left[-5e^{-t} + 3e^{-3t} + 2\cos t + 4\sin t\right]$$

Similarly,

$$I_1 = \frac{2s+3}{s} I_2 = \frac{E}{3}\left[\frac{2s+3}{(s+1)(s+3)(s^2+1)}\right]$$
$$= \frac{E}{60}\left[\frac{5}{s+1} + \frac{3}{s+3} - \frac{8s}{s^2+1} + \frac{14}{s^2+1}\right]$$

and

$$i_1 = \frac{E}{60}\left[5e^{-t} + 3e^{-3t} - 8\cos t + 14\sin t\right]$$

2. Convolution

The convolution of two functions $f(t)$ and $g(t)$, denoted $f * g$ is defined by

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du$$

Convolution has many of the properties of a product, but not all. For example, it is commutative: $f * g = g * f$ since

$$\begin{aligned}(f * g)(t) &= \int_0^t f(u)g(t-u)\,du \quad \text{and with } v = t-u \\ &= \int_t^0 f(t-v)g(v)\,(-dv) = \int_0^t g(v)f(t-v)\,dv = (g * f)(t)\end{aligned}$$

However, note that $1 * 1 \neq 1$, since

$$1 * 1 = \int_0^t du = t$$

Also, $f * f$ can be negative, e.g., if $f(t) = \cos t$

$$\begin{aligned}\cos t * \cos t &= \int_0^t \cos u \cos(t-u)\,du \\ &= \frac{1}{2}\int_0^t \left[\cos t + \cos(2u-t)\right]\,du \\ &= \frac{1}{2}\left[u\cos t + \frac{1}{2}\sin(2u-t)\right]_0^t \\ &= \frac{1}{2}\left\{t\cos t + \frac{1}{2}\sin t - \frac{1}{2}\sin(-t)\right\} \\ &= \frac{1}{2}\left[t\cos t + \sin t\right]\end{aligned}$$

This is negative at, for example, $t = 3\pi/2$.

The value of convolutions in the context of Laplace transforms arises from the following:

THEOREM 2.1. *If $f(t)$ and $g(t)$ have Laplace transforms $F(s)$ and $G(s)$ respectively, then*

$$\mathcal{L}(f * g)(t) = F(s)G(s)$$

Proof.

$$\begin{aligned}F(s)G(s) &= \left[\int_0^\infty e^{-su}f(u)\,du\right]\left[\int_0^\infty e^{-sv}f(v)\,dv\right] \\ &= \int_0^\infty \left[\int_0^\infty e^{-s(u+v)}f(u)g(v)\,dv\right]\,du\end{aligned}$$

Now, let $t = u + v$ and so $dt = dv$ (in the inner integral). Then

$$\begin{aligned}F(s)G(s) &= \int_0^\infty \left[\int_u^\infty e^{-st}f(u)g(t-u)\,dt\right]\,du \\ &= \int_0^\infty \left[\int_0^t e^{-st}f(u)g(t-u)\,du\right]\,dt\end{aligned}$$

(switching the order of integration by taking note of the area of integration)

Thus

$$F(s)G(s)) = \int_0^\infty e^{-st} \left[\int_0^t f(u)g(t-u)\,du \right] dt$$
$$= \int_0^\infty e^{-st} (f * g)(t)\,dt$$
$$= \mathcal{L}(f * g)(t)$$

Example. $f(t) = g(t) = \cos t \Rightarrow$

$$F(s) = G(s) = \frac{s}{s^2+1}$$

and so

$$F(s)G(s) = \frac{s^2}{(s^2+1)^2}$$

Thus

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+1)^2}\right] = \cos t * \cos t = \frac{1}{2}\left[t\cos t + \sin t\right]$$

Note that we can arrive at this result in an alternative fashion by using

$$\mathcal{L}[t\cos t] = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$
$$= \frac{-(s^2+1)+s(2s)}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

Thus

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{\frac{1}{2}(s^2-1)+\frac{1}{2}(s^2+1)}{(s^2+1)^2}\right]$$
$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s^2-1}{(s^2+1)^2} + \frac{1}{s^2+1}\right]$$
$$= \frac{1}{2}\left[t\cos t + \sin t\right]$$

(as before)

2.1. Initial value problems. Convolution can obviously be used as an alternative way of solving differential equations. One advantage can be seen in the following:

Example.

$$y'' + \omega^2 y = h(t) \qquad y(0) = A, \quad y'(0) = B$$

Transforming $\Rightarrow$

$$s^2Y - As - B + \omega^2 Y = H(s)$$

and thus

$$(s^2+\omega^2)Y = H(s) + As + B$$

or

$$Y = \frac{H(s)}{s^2+\omega^2} + \frac{As}{s^2+\omega^2} + \frac{B}{s^2+\omega^2}$$

and thus

$$y = \frac{1}{\omega}\sin\omega t * h(t) + A\cos\omega t + \frac{B}{\omega}\sin\omega t$$

Thus we can obtain a solution to a problem in this general form using convolutions without knowing $h(t)$ and if we do know $h(t)$ we can use this formula to calculate y without having to transform r.

2.2. Integral equations. Convolutions can also be used together with Laplace transforms to solve some integral equations very simply.

Example.

$$y(t) = \sin 2t + \int_0^t y(u) \sin 2(t-u)\, du$$

Transforming $\Rightarrow$

$$\begin{aligned} Y &= \frac{2}{s^2+2^2} + \frac{2Y}{s^2+2^2} \\ \Rightarrow \quad \left[1 - \frac{2}{s^2+4}\right] Y &= \frac{2}{s^2+4} \\ \Rightarrow \quad Y = \frac{s^2+4}{s^2+2}\left(\frac{2}{s^2+4}\right) & \\ &= \frac{2}{s^2+2} \end{aligned}$$

and thus $y(t) = \sqrt{2} \sin \sqrt{2} t$.