UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4005

SEMESTER: Autumn 2016/17

MODULE TITLE: Engineering Mathematics 3

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**
A table of *Laplace* transforms is attached.

Table of Laplace Transforms

$f(t),\ t \geq 0$	$F(s) = \mathcal{L}\left(f(t)\right) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{1}{a-b}(e^{at} - e^{bt})$	$\frac{1}{(s-a)(s-b)}$
$\frac{a}{a-b}e^{at} - \frac{b}{a-b}e^{bt}$	$\frac{s}{(s-a)(s-b)}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\int_0^t f(\tau)\, d\tau$	$\frac{1}{s}F(s)$
$e^{at}f(t)$	$F(s-a)$
Heaviside $u_a(t)$	$\frac{e^{-as}}{s}$
$f(t-a)u_a(t)$	$e^{-as}F(s)$
Ramp $R(t-a)$	$\frac{e^{-as}}{s^2}$
$tf(t)$	$-F'(s)$
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(\sigma)\, d\sigma$
$(f * g)(t) \equiv \displaystyle\int_0^t f(t-\tau)g(\tau)\, d\tau$	$F(s)G(s)$
$f(t) = f(t+p)$	$\dfrac{1}{1-e^{-sp}} \displaystyle\int_0^p f(t)e^{-st}\, dt$

1. (a) Sketch the graphs of the following functions $f(t), t \geq 0$ and find their *Laplace* transforms:

   (i) $f(t) = \begin{cases} 0, & \text{if } t < 5 \\ e^{-2t+10}, & \text{if } 5 \leq t \end{cases}$ 2

   (ii) $f(t) = \begin{cases} \sin 3t, & \text{if } 0 \leq t < 2\pi \\ 0, & \text{otherwise} \end{cases}$ 3

   (iii) $f(t) = \begin{cases} 2t, & \text{if } 0 \leq t < 1 \\ 2 - t, & \text{if } 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$ 3

   (b) Find the inverse *Laplace* transforms of the following 2,3,3

   $$(i)\ \frac{s+2}{s^2+4s+3} \qquad (ii)\ \frac{e^{-3s}}{s^2+4} \qquad (iii)\ \arctan\left(\frac{s+1}{s-1}\right)$$

2. (a) Use *Laplace* transforms to solve the initial value problem 10

   $$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 6t - 1, \qquad y(0) = 1, \quad \frac{dy}{dt}(0) = -2$$

   (b) Use *Laplace* transforms to solve the integral equation 6

   $$y(t) = e^t - \int_0^t e^{-(t-u)} y(u)\, du$$

3. (a) Sketch the graph of the following periodic function

   $$f(x) = \begin{cases} x + \pi, & \text{if } -\pi \leq x < 0 \\ x - \pi, & \text{if } 0 \leq x < \pi \end{cases} \qquad f(x + 2\pi) = f(x)$$

   and show that its *Fourier* series is given by

   $$-2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

   6

   (b) Hence evaluate the series 4

   $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

   (c) Use the *Fourier* series of part (a) to find a particular solution of the differential equation 6

   $$\frac{d^2y}{dx^2} + 2y = f(x)$$

   where $f(x)$ is as in part (a).

4. (a) Let P_3 be the space of polynomials of degree at most three, i.e. $P_3 = \{p(x) : p(x) = a_0 + a_1x + a_2x^2 + a_3x^3\}$ where $a_0,\ a_1,\ a_2$ and a_3 are real numbers. Determine whether the set of polynomials

$$\{1 + x,\ 2 - 2x + x^2,\ 1 + 3x - (1/2)x^2 + 2x^3,\ 2x + x^3\}$$

forms a basis for P_3 6

(b) Find bases for the row space and column space, and hence the rank, of the matrix 6

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 5 & 18 \\ 3 & -3 & 6 \\ 0 & 1 & 2 \end{pmatrix}$$

Does the system $Ax = b$ have a solution when

$$b = \begin{pmatrix} 6 \\ -6 \\ 12 \\ -2 \end{pmatrix}$$

and, if so, find it. 4

5. (a) Let P_2 be the space of polynomials of degree at most two. Use the *Gram-Schmidt* process to transform the standard basis $\{1,\ x,\ x^2\}$ for P_2 to the orthonormal one defined by the inner product

$$< p, q >= \sum_{i=1}^{5} p(x_i)q(x_i)$$

where $x_1 = -2,\ x_2 = -1,\ x_3 = 0,\ x_4 = 1,\ x_5 = 2.$ 8

(b) Let f be a function on $[-2, 2]$, taking the value $f(x_i)$ at $x = x_i,\ i = 1, 2, 3, 4, 5$ as given in the following table

x_i	$f(x_i)$
-2	4
-1	-2
0	-2
1	-1
2	5

Find the least squares approximation to f in P_2 using the inner product defined in part (a). 8

6. Find the eigenvalues and corresponding eigenspaces of the matrix 8

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}.$$

Show that A is diagonalisable over the reals and find the matrix P that diagonalises it. 4

Hence or otherwise find the matrix exponential e^{At}. 4

7. (a) Find the LU decomposition ($A = LU$), where L is a unit lower triangular matrix and U an upper triangular matrix of the matrix 5

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -3 & -2 \\ -1 & 9 & 12 \end{pmatrix}$$

and hence solve the system of equations $Ax = b$ where 3

$$b = \begin{pmatrix} 3 \\ 4 \\ -8 \end{pmatrix}$$

(b) The system of equations $Ax = b$ can also be solved using an iterative scheme. One such scheme for a square matrix A is the *Jacobi* method in which successive approximations are generated by

$$x_{k+1} = D^{-1}b - D^{-1}\tilde{A}x_k$$

where x_k is the k-th approximation, and $A = D + \tilde{A}$ with D being a diagonal matrix whose entries are the respective diagonal entries of A, and $\tilde{A} = A - D$. Show how the iteration scheme may be derived. Under what condition does the scheme converge for any initial approximation x_0 ? 4

Using the fact that the roots of the equation

$$\lambda^3 - (3/4)\lambda - (7/36) = 0$$

satisfy $|\lambda| < 1$, does the scheme converge for the A matrix of part (a)? Explain your answer. 4