A periodic function $f(x)$ of period $2L$ has *Fourier* Series (FS) given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots \end{aligned}$$

1. (a) Period is 4 ($L = 2$). Function is odd. Hence

$$a_n = 0$$

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \\ &= \int_0^2 x\sin\left(\frac{n\pi x}{2}\right) dx \\ &= \int_0^2 x\, d\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right) \\ &= -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx \\ &= -\frac{4}{n\pi}\cos(n\pi) + 0 + \frac{2}{n\pi}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right)\Big|_0^2 \\ &= \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{(n\pi)^2}\left(\sin(n\pi) - \sin 0\right) \\ &= \frac{4}{n\pi}(-1)^{n+1} \end{aligned}$$

The FS is

$$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)$$

(b) Period is π ($L = \frac{\pi}{2}$). Function is even. Hence

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin x\cos(2nx)dx \end{aligned}$$

Using the trignometric identity

$$\sin A\cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

gives

$$\begin{aligned} a_n &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin((1+2n)x) + \sin((1-2n)x)}{2}dx \\ &= \frac{2}{\pi}\left(-\frac{1}{1+2n}\cos((1+2n)x) - \frac{1}{1-2n}\cos((1-2n)x)\right)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{2}{\pi}\left(-\frac{1}{1+2n}\left(\cos(\tfrac{\pi}{2}+n\pi) - \cos(0)\right) - \frac{1}{1-2n}\left(\cos(\tfrac{\pi}{2}-n\pi) - \cos(0)\right)\right) \\ &= \frac{2}{\pi}\left(\frac{1}{1+2n} + \frac{1}{1-2n}\right) \\ &= \frac{4}{\pi}\frac{1}{1-4n^2} \end{aligned}$$

The FS is

$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{1-4n^2}\cos(2nx)$$

(c) Period is 2 ($L = 1$). Hence

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \int_{-1}^{1} f(x)\cos(n\pi x)dx \\
&= \int_{-1}^{0} 0dx + \int_{0}^{1} x^2\cos(n\pi x)dx
\end{aligned}$$

if $n \neq 0$

$$\begin{aligned}
&= \int_0^1 x^2 d\left(\frac{1}{n\pi}\sin(n\pi x)\right) \\
&= \frac{x^2\sin(n\pi x)}{n\pi}\bigg|_0^1 - \frac{2}{n\pi}\int_0^1 x\sin(n\pi x)dx \\
&= 0 - 0 - \frac{2}{n\pi}\int_0^1 xd\left(-\frac{1}{n\pi}\cos(n\pi x)\right) \\
&= \frac{2}{(n\pi)^2}\left(x\cos(n\pi x)\big|_0^1 - \int_0^1 \cos(n\pi x)dx\right) \\
&= \frac{2}{(n\pi)^2}\left(\cos(n\pi) - 0 - \frac{1}{n\pi}\sin(n\pi x)\big|_0^1\right) \\
&= \frac{2}{(n\pi)^2}\left((-1)^n - \frac{1}{n\pi}(0-0)\right) \\
&= \frac{2}{(n\pi)^2}(-1)^n
\end{aligned}$$

if $n = 0$

$$\begin{aligned}
&= \frac{x^3}{3}\bigg|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)dx \\
&= \int_{-1}^{0} 0dx + \int_0^1 x^2\sin(n\pi x)dx \\
&= \int_0^1 x^2 d\left(-\frac{1}{n\pi}\cos(n\pi x)\right) \\
&= -\frac{x^2\cos(n\pi x)}{n\pi}\bigg|_0^1 + \frac{2}{n\pi}\int_0^1 x\cos(n\pi x)dx \\
&= -\frac{\cos(n\pi)}{n\pi} + 0 + \frac{2}{n\pi}\int_0^1 xd\left(\frac{1}{n\pi}\sin(n\pi x)\right) \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{(n\pi)^2}\left(x\sin(n\pi x)\big|_0^1 - \int_0^1 \sin(n\pi x)dx\right) \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{(n\pi)^2}\left(\sin(n\pi) - 0 + \frac{1}{n\pi}\cos(n\pi x)\big|_0^1\right) \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{2}{(n\pi)^3}(\cos(n\pi) - \cos 0)) \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{2((-1)^n - 1)}{(n\pi)^3}
\end{aligned}$$

The FS is

$$\frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) - \frac{4}{\pi^3}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^3}\sin(n\pi x)$$

2. The even extension is

$$f(x) = x^2, \quad \text{if } \; -1 \le x \le 1$$

Here $L = 1$ and hence

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= 2\int_0^1 x^2\cos(n\pi x)dx \\ &= \begin{cases} \frac{4}{(n\pi)^2}(-1)^n, & \text{if } n \ne 0 \\ \frac{2}{3}, & \text{if } n = 0 \end{cases} \qquad \text{See Q1(c)} \end{aligned}$$

The Cosine Series is

$$\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)$$

The odd extension is

$$f(x) = \begin{cases} -x^2, & \text{if } -1 < x < 0 \\ , x^2 & \text{if } 0 \le x \le 1, \end{cases}$$

Here $L = 1$ and hence

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= 2\int_0^1 x^2\cos(n\pi x)dx \\ &= \frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{(n\pi)^3} \qquad \text{See Q1(c)} \end{aligned}$$

The Sine Series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x) - \frac{8}{\pi^3}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^3}\sin(n\pi x)$$

3. (a) Consider the FS of Q1(a). At $x = 1$ where $f(x)$ is continuous, we have

$$\begin{aligned} 1 &= \frac{4}{\pi}\left(\sin(\pi/2) - (1/2)\sin(\pi) + (1/3)\sin(3\pi/2) - (1/4)\sin(2\pi) + \cdots\right) \\ &= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) \\ &= \frac{4}{\pi}\left(\frac{2}{1\times 3} + \frac{2}{5\times 7} + \cdots\right) \end{aligned}$$

Hence

$$\frac{1}{1\times 3} + \frac{1}{5\times 7} + \cdots = \frac{\pi}{8}$$

(b) Consider the Cosine Series of Q2. At $x = 0$ where $f(x)$ is continuous, we have

$$\begin{aligned} 0 &= \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \\ &= \frac{1}{3} - \frac{4}{\pi^2}\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\right) \end{aligned}$$

Hence

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}$$

(c) From the Cosine Series of Q2, at $x = 1$, where $f(x)$ is continuous, we have

$$\begin{aligned} 1 &= \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\ &= \frac{1}{3} + \frac{4}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) \end{aligned}$$

Hence

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

4. (a) The given $f(x)$ is odd with $L = 1$. Hence

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= 2\int_0^1 \sin(n\pi x)dx \\ &= -2\frac{1}{n\pi}\cos(n\pi x)\Big|_0^1 \\ &= -\frac{2}{n\pi}(\cos n\pi - \cos 0) \\ &= \frac{2}{n\pi}(1-(-1)^n) \end{aligned}$$

Its FS is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(n\pi x)$$

The DE is now

$$\frac{dy}{dx} + 2y = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(n\pi x)$$

Let y_n^p be the particular solution corresponding to the forcing function component $b_n\sin(n\pi x)$. By the *Method of Undetermined Coefficients*, we have

$$\begin{aligned} y_n^p &= A_n\cos(n\pi x) + B_n\sin(n\pi x) \\ \Rightarrow \frac{dy_n^p}{dx} &= -n\pi A_n\sin(n\pi x) + n\pi B_n\cos(n\pi x) \end{aligned}$$

Substituting this into the DE gives

$$\begin{aligned} -n\pi A_n\sin(n\pi x) + n\pi B_n\cos(n\pi x) + 2\left(A_n\cos(n\pi x) + B_n\sin(n\pi x)\right) &= b_n\sin(n\pi x) \\ \Rightarrow (2A_n + n\pi B_n)\cos(n\pi x) + (-n\pi A_n + 2B_n)\sin(n\pi x) &= 0\cos(n\pi x) + b_n\sin(n\pi x) \end{aligned}$$

which gives the simultaneous equations

$$\begin{aligned} 2A_n + n\pi B_n &= 0 \\ -n\pi A_n + 2B_n &= b_n \end{aligned}$$

which have solutions

$$A_n = \frac{-n\pi b_n}{(n\pi)^2+4} \qquad B_n = \frac{2b_n}{(n\pi)^2+4}$$

where b_n is as determined above.
Thus

$$\begin{aligned} y^p &= \sum_{n=1}^{\infty} y_n^p \\ &= \sum_{n=1}^{\infty}\left(A_n\cos(n\pi x) + B_n\sin(n\pi x)\right) \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\left(-\frac{\pi(1-(-1)^n)}{(n\pi)^2+4}\cos(n\pi x) + \frac{2(1-(-1)^n)}{(n\pi)^2+4}\sin(n\pi x)\right) \end{aligned}$$

(b) To simplify computations(?), let us take the odd extension of $f(x)$; then $L = 2$ and

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \int_0^2 f(x)\sin(n\pi x/2)dx \\ &= \int_0^1 x\sin(n\pi x/2)dx + \int_1^2 (2-x)\sin(n\pi x/2)dx \end{aligned}$$

Use a change of variable in the second integral (e.g. $u = 2 - x$) to convert this to

$$
\begin{aligned}
b_n &= (1-(-1)^n)\int_0^1 x\sin(n\pi x/2)dx \\
&= (1-(-1)^n)\int_0^1 xd\left(-\frac{2}{n\pi}\cos(n\pi x/2)\right) \\
&= \frac{2(1-(-1)^n)}{n\pi}\left(-x\cos(n\pi x/2)\Big|_0^1 + \int_0^1 \cos(n\pi x/2)dx\right) \\
&= \frac{2(1-(-1)^n)}{n\pi}\left(-\cos(n\pi/2) + 0 + \frac{2}{n\pi}\sin(n\pi x/2)\Big|_0^1\right) \\
&= \frac{2(1-(-1)^n)}{n\pi}\left(-\cos(n\pi/2) + \frac{2}{n\pi}\sin(n\pi/2)\right) \\
&= \frac{8}{(n\pi)^2}\sin(n\pi/2) \qquad\qquad !!
\end{aligned}
$$

To see where this last line comes from, consider the cases n even and n odd separately.
The DE is now

$$\frac{d^2y}{dx^2} + y = \sum_{n=1}^{\infty} b_n \sin(n\pi x/2)$$

Let y_n^p be the particular solution corresponding to the n-th harmonic: then $y_n^p = A_n\cos(n\pi x/2) + B_n\sin(n\pi x/2)$ and the DE becomes

$$-(n\pi/2)^2A_n\cos(n\pi x/2) - (n\pi/2)^2B_n\sin(n\pi x/2) + A_n\cos(n\pi x/2) + B_n\sin(n\pi x/2) = b_n\sin(n\pi x/2)$$

$$\Rightarrow (1-(n\pi/2)^2)A_n\cos(n\pi x/2) + (1-(n\pi/2)^2B_n\sin(n\pi x/2) = 0\cos(n\pi x/2) + b_n\sin(n\pi x/2)$$

.....yielding the simultaneous equations

$$
\begin{aligned}
(1-(n\pi/2)^2)A_n &= 0 \\
(1-(n\pi/2)^2B_n &= b_n
\end{aligned}
$$

which have solutions

$$A_n = 0 \qquad B_n = \frac{b_n}{1-(n\pi/2)^2}$$

where b_n is as determined above.
Thus

$$
\begin{aligned}
y^p &= \sum_{n=1}^{\infty} y_n^p \\
&= \sum_{n=1}^{\infty}(A_n\cos(n\pi x/2) + B_n\sin(n\pi x/2)) \\
&= \sum_{n=1}^{\infty}\frac{8\sin(n\pi/2)}{(n\pi)^2(1-(n\pi/2)^2}\sin(n\pi x/2)
\end{aligned}
$$