1. To test whether a subset of a vector space is a subspace, it is sufficient to test whether it is closed under the operations of (vector) addition and (scalar) multiplication. In this case where the vector space is the set of real valued functions of a real variable, let $f$ and $g$ be appropriate functions and $\alpha \in \mathbf{R}$ a scalar.

   (a) Set is a subspace since (i) it is closed under addition

   $$f(0) + g(0) = 0 + 0 = 0$$

   and (ii) closed under multiplication

   $$\alpha f(0) = \alpha 0 = 0$$

   (b) Set is **not** a subspace since even though (i) it is closed under addition

   $$f(x) + g(x) \leq 0 + 0 = 0$$

   (ii) it is not closed under multiplication: take $\alpha < 0$ then

   $$\alpha f(x) \geq 0$$

   (c) Set is **not** a subspace. Again it is closed under addition: $f(x) + g(x)$ is a polynomial of degree at most 3 with integer coefficients; but it is not closed under multiplication: take $\alpha$ as an irrational number e.g. $\sqrt{2}$, then $\alpha f(x)$ is a polynomial of degree at most 3 but it does not have integer coefficients.

2. (a) No. For instance

   $$\begin{pmatrix} -1 \\ 10 \\ 8 \\ 6 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ 2 \\ -4 \\ 6 \end{pmatrix}$$

   Formally the solution of

   $$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -5 \\ 2 \\ -4 \\ 6 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 \\ 10 \\ 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   which is equivalent to solving the system of equations

   $$\begin{pmatrix} 1 & -5 & -1 \\ 2 & 2 & 10 \\ 3 & -4 & 8 \\ 0 & 6 & 6 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

   can be found using *Gaussian* Elimination. For instance using the *augmented matrix* form

   $$\begin{bmatrix} 1 & -5 & -1 & \vdots & 0 \\ 2 & 2 & 10 & \vdots & 0 \\ 3 & -4 & 8 & \vdots & 0 \\ 0 & 6 & 6 & \vdots & 0 \end{bmatrix}$$

we get on the first pass

$$\left[\begin{array}{ccc:c} 1 & -5 & -1 & 0 \\ 0 & 12 & 12 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & 6 & 6 & 0 \end{array}\right]$$

and on the second pass

$$\left[\begin{array}{ccc:c} 1 & -5 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which yields the solution

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix} t \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for arbitrary scalar $t \neq 0$. (The choice $t = 1$ corresponds to relationship between the three vectors given at the start).

(b) Since $S$ does not span $\mathbf{R}^4$, an arbitrary vector can not be represented as a linear combination of the vectors of $S$. However the given vector $(7, 2, 10, -6)^T$ can be !
From part(a), we don't have to use all 3 of the vectors of $S$ (why ?) to find the required representation. If we use the first two vectors, we are trying to solve

$$\begin{pmatrix} 7 \\ 2 \\ 10 \\ -6 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -5 \\ 2 \\ -4 \\ 6 \end{pmatrix}$$

or (using the augmented matrix approach )

$$\left[\begin{array}{cc:c} 1 & -5 & 7 \\ 2 & 2 & 2 \\ 3 & -4 & 10 \\ 0 & 6 & -6 \end{array}\right]$$

which has reduced row echelon form

$$\left[\begin{array}{cc:c} 1 & -5 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The non trivial part is equivalent to the set of equations

$$\begin{aligned} \alpha_1 - 5\alpha_2 &= 7 \\ \alpha_2 &= -1 \end{aligned}$$

which has solution

$$\alpha_1 = 2, \qquad \alpha_2 = -1$$

Hence

$$\begin{pmatrix} 7 \\ 2 \\ 10 \\ -6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -5 \\ 2 \\ -4 \\ 6 \end{pmatrix}$$

3. (a) $n+1$ (See next part)

(b) $\{1, x, x^2, \ldots, x^n\}$ is linearly independent since the only solution of

$$\alpha_1(1) + \alpha_2(x) + \alpha_3(x^2) + \ldots + \alpha_{n+1}(x^n) = 0$$

is obviously

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_{n+1} = 0$$

(Alternatively, no power of $x$ can be expressed as a linear combination of the other powers of $x$ for arbitrary values of $x$.) Additionally, the set spans $P_n$ as any polynomial of degree at most $n$ can be expressed as a linear combination of the functions in the set in an obvious way.

(c) To check whether the given set spans $P_2$, solving

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 &= \alpha_1(1+x) + \alpha_2(2+x) + \alpha_3(2-x^2) \\ &= (\alpha_1 + 2\alpha_2 + 2\alpha_3) + (\alpha_1 + \alpha_2)x + (-\alpha_3)x^2 \end{aligned}$$

gives

$$\begin{aligned} \alpha_1 &= -a_0 + 2a_1 - 2a_2 \\ \alpha_2 &= a_0 - a_1 + 2a_2 \\ \alpha_3 &= -a_2 \end{aligned}$$

To check the linear independence of the given set, setting $a_0 = a_1 = a_2 = 0$ in the above derivation gives $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
For the polynomial $x^2 + 1$, the coefficients of the linear combination are $\alpha_1 = -3, \alpha_2 = 3, \alpha_3 = -1$ respectively, or $1 + x^2 = -3(1+x) + 3(2+x) - (2-x^2)$.

4. (a) Obtaining the reduced row echelon form of the matrix using *Gaussian* Elimination gives

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 1 & 5 & 7 & 0 \\ 0 & 0 & 3 & -2 \\ 1 & 0 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 3 & 6 & -4 \\ 0 & 0 & 3 & -2 \\ 0 & -2 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 6 & -4 \\ 0 & 0 & 3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & -4 \\ 0 & 0 & 3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank = 3

(b) The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 4 & \vdots & 17 \\ 1 & 5 & 7 & 0 & \vdots & -24 \\ 0 & 0 & 3 & -2 & \vdots & -19 \\ 1 & 0 & 1 & 4 & \vdots & 19 \end{pmatrix}$$

Using the same elementary row operations as in part(a) yields

$$\begin{pmatrix} 1 & 2 & 1 & 4 & \vdots & 17 \\ 0 & 3 & 6 & -4 & \vdots & -41 \\ 0 & 0 & 3 & -2 & \vdots & -19 \\ 0 & -2 & 0 & 0 & \vdots & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 & \vdots & 17 \\ 0 & 1 & 0 & 0 & \vdots & 2 \\ 0 & 3 & 6 & -4 & \vdots & -41 \\ 0 & 0 & 3 & -2 & \vdots & -19 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 & \vdots & 17 \\ 0 & 1 & 0 & 0 & \vdots & -1 \\ 0 & 0 & 6 & -4 & \vdots & -38 \\ 0 & 0 & 3 & -2 & \vdots & -19 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 & \vdots & 17 \\ 0 & 1 & 0 & 0 & \vdots & -1 \\ 0 & 0 & 3 & -2 & \vdots & -19 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

The nontrivial part is equivalent to the equations

$$\begin{aligned} w + 2x + y + 4z &= 17 \\ x &= -1 \\ 3y - 2z &= -19 \end{aligned}$$

which yields (letting $z = t$)

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (76-14t)/3 \\ -1 \\ (-19+2t)/3 \\ t \end{pmatrix} = \begin{pmatrix} 76/3 \\ -1 \\ -19/3 \\ 0 \end{pmatrix} + \begin{pmatrix} -14/3 \\ 0 \\ 2/3 \\ 1 \end{pmatrix} t$$