Maths Enrichment: More Sequences

Let $\{a_n\}$ and $\{b_n\}$represent sequences, and let $S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$

1. If $a_1 = 1$ and $a_n = a_{n-1} + \frac{1}{a_{n-1}}$, $n > 1$, prove that $a_{100} > 14$.
   Hint: Find a linear recurrence inequality that $b_n = a_n^2$ satisfies and solve this.

2. If $a_1 = 1$ and $a_n = \frac{n+1}{n-1}(a_1 + a_2 + \ldots + a_{n-1})$, $n > 1$, find a_{2010}.
   Hint: Find a linear recurrence that $b_n = \frac{S_n}{n}$ satisfies.

3. If $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$, $n > 1$,
   (a) show that $S_n = a_{n+2} - 1$
   (b) Evaluate
   $$\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots$$
   Hint: Find a linear recurrence that $b_n = \frac{a_n}{2^n}$ satisfies and then add up the terms on each side of the recurrence as n goes from 1 to infinity.

4. Consider the sequence of points in the plane
   $$T_1 = (1, 0) \quad T_2 = (0, 1) \quad T_3 = (-1, 0)$$
   $$T_{n+3} = \text{midpoint of line segment} \overline{T_n T_{n+1}}$$
   If it exists, find $\lim_{n\to\infty} T_n$.
   Hint: By considering $T_n = (x_n, y_n)$ show that $T_n + 2T_{n+1} + 2T_{n+2}$ is constant, and then take limits as n goes to infinity.

5. Find the limit of the sequence
   $$\sqrt{13}, \quad \sqrt{13 - \sqrt{13}}, \quad \sqrt{13 - \sqrt{13 + \sqrt{13}}}, \quad \sqrt{13 - \sqrt{13 + \sqrt{13 - \sqrt{13}}}}, \ldots$$
   Hint: Find a recurrence that relates a_n to a_{n-2}, and take limits as n goes to infinity.

6. $a_1, a_2, \ldots, a_{2010}$ are positive real numbers which satisfy
   $$a_1 = a_{2010}, \qquad a_{n-1} + \frac{2}{a_{n-1}} = 2a_n + \frac{1}{a_n}$$
   Find the maximum value that a_1 can have.
   Hint: Find the two possible recurrences that relate a_n to a_{n-1}. Use both of these to determine the possible values of a_n starting from a_1.

7. Consider the three sequences defined by
   $$\begin{aligned} a_{n+1} &= b_n + \frac{1}{c_n}, & a_1 > 0 \\ b_{n+1} &= c_n + \frac{1}{a_n}, & b_1 > 0 \\ c_{n+1} &= a_n + \frac{1}{b_n}, & c_1 > 0 \end{aligned}$$
   Prove that none of the three sequences is bounded.
   Hint: Find a recurrence inequality for $t_n = a_n + b_n + c_n$. You will need to invoke the well known inequality $\frac{1}{p+q} \leq \frac{1}{p} + \frac{1}{q}$, where p and q are positive numbers.