UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018

SEMESTER: Spring 2009

MODULE TITLE: Dynamical Systems

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 100 %

EXTERNAL EXAMINER: Prof. J. Flavin

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time vector sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

1. The following dynamical system represents the interaction between a prey population and a predator population of sizes $x(t),\ y(t) \geq 0$ respectively in some normalised units.

$$\dot{x} = 2x\left(1 - \frac{x}{100}\right) - y\left(\frac{x}{30+x}\right), \qquad \dot{y} = y\left(-1 + 2\left(\frac{x}{30+x}\right)\right)$$

   (a) Find the fixed points of the system. 4

   (b) Use linearisation to characterise the nature of the fixed points. 8

   (c) Show that $y = 0$ and $x = 0$ are trajectories of the system. 2

   (d) Can the system exhibit closed orbits? Explain your answer. 6

2. Consider the traffic model, defined on the interval $0 \leq x \leq 1$, by the cubic map

$$x' = \lambda x(1 - x^2)$$

   where $\lambda > 0$.

   (a) Find the fixed points of the map and determine their stability properties as a function of the parameter λ. 8

   (b) Show that there is a two-cycle consisting of the points $x_1,\ x_2$ where

$$x_{1,2} = \sqrt{\frac{1 \pm \sqrt{1 - (2/\lambda)^2}}{2}}$$

   whenever $\lambda > 2$.
   Further show that this cycle is stable for $2 < \lambda < \sqrt{5}$. 12

3. (a) For the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   where the origin is a saddle point, define the stable and unstable manifolds. 4

   (b) Compute the stable and unstable manifolds of the linear system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix} \mathbf{x}$$

   8

(c) The system

$$\dot{x} = y - \frac{3}{25}(4x^2 + 3xy - y^2), \qquad \dot{y} = 4x + 3y - \frac{17}{25}(4x^2 + 3xy - y^2)$$

has a trajectory given by

$$y = 4x$$

Is this orbit an invariant manifold, and if so, to which manifold does it correspond? 8

4. (a) Define the centre manifold of a fixed point of a map. 4

(b) Show that the system

$$x' = -x + xy, \qquad y' = x^2$$

has a non-hyperbolic fixed point at the origin. In a neighbourhood of the origin, the centre manifold can be written as $y = h(x)$. Show how a power series approximation to the function h may be computed, and compute the first two non-zero terms of the power series of the centre manifold. 8

(c) Use the centre manifold calculated in part (b) to investigate the local stability properties of the origin. 8

5. (a) State the *Poincaré-Bendixson* theorem. 6

(b) The *Kaldor* business cycle model is

$$\dot{Y} = \alpha(I - S), \qquad \dot{K} = I - \delta K$$

where Y is income, I investment, S savings, K capital, α an adjustment factor and δ the capital depreciation rate. In addition, assume that

$$I = A(B + \tanh(Y - Y_c)) - \beta K, \qquad S = \gamma Y$$

and the parameter values are $\alpha = 1.0$, $\delta = 0.05$, $A = 1.1$, $B = 2.4$, $Y_c = 2.2$, $\beta = 0.55$ and $\gamma = 0.1$. Use the *Poincaré-Bendixson* theorem to prove that the model exhibits a limit cycle. 14

6. The equation

$$\frac{dN}{dt} = rN(N-A)(1-\frac{N}{K}) + MN$$

where $r,\ K > A > 0$ and $M \geq 0$ is a model of the population growth dynamics of a "fragile" species, i.e. one liable to becoming extinct, that is being managed by the introduction of extra members of the species from elsewhere. $N(t)$ is the population size at time t, r is the intrinsic growth rate, K is the carrying capacity, A is the population threshold below which the species tends to extinction and MN is the number of introduced conspecifics expressed as a proportion M of population size.

(a) Show that the system can be rescaled as

$$\frac{dx}{d\tau} = x(x-a)(1-x) + mx$$

for suitably defined dimensionless quantities $x,\ \tau,\ a$ and m. Note that $a < 1$. 6

(b) In the case of no management ($m = 0$), show that the population goes to extinction if $x < a$. 6

(c) Find the fixed points of the system of part (a) as a function of the parameter m, and show that a bifurcation occurs; classify it and determine the critical value $m = m_c$. 6

(d) How can extinction be avoided? 2

7. (a) State the *Neimark-Sacker* bifurcation theorem for a two-dimensional map. 4

(b) Let

$$x' = x + y, \qquad y' = r - x^2$$

where $r > 0$.
Find the fixed points of the system as functions of the parameter r and classify them. 8

(c) For the system of part (b), use the *Neimark-Sacker* theorem to show that one of the fixed points undergoes a *Neimark-Sacker* bifurcation at a certain value of the parameter $r = r_c$. Find the value r_c. 8