UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018

SEMESTER: Spring 2011

MODULE TITLE: Dynamical Systems

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 100 %

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time vector sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

1. The following dynamical system represents a mutualistic interaction between two self-limiting species with normalised population densities of sizes $x(t),\ y(t) \geq 0$ respectively.

$$\dot{x} = x\,(2 + y - 3x)\,, \qquad \dot{y} = y\,(1 + 2x - y)$$

   (a) Find the fixed points of the system. 4

   (b) Use linearisation to characterise the nature of the fixed points. 8

   (c) Can the system exhibit closed orbits ? Explain your answer. 8

2. Consider the stylised economic growth model described by the 1-parameter map defined on the interval $0 \leq x \leq 1$ by

$$x' = \alpha x(1 - \sqrt{x})$$

   where x represents capital accumulated in any given year and $\alpha > 0$ is a growth parameter.

   (a) Show that $\alpha \leq 27/4$ to ensure that the orbit of x remains in $[0,\ 1]$ for all initial values in this interval. 4

   (b) Find the fixed points of the map, and determine their stability properties as a function of the parameter α. 8

   (c) What happens as α passes through values (i) $\alpha = 1$ and (ii) $\alpha = 5$? 4

   (d) It can be shown that there are no period-2 points for $\alpha < 5$. What can be said about other periodic orbits for $\alpha < 5$? 4

3. (a) For the dynamical system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   where the origin is a saddle point, define the stable and unstable manifolds. 4

   (b) Compute the stable and unstable manifolds of the linear system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ \frac{4}{9} & 1 \end{pmatrix} \mathbf{x}$$

   8

(c) The system

$$x' = y, \qquad y' = \frac{4}{9}x + y + \frac{1}{4}x^2 + \frac{3}{4}y^2$$

with a saddle point at the origin, has an invariant manifold which may be written as $y = h(x)$ in a neighbourhood of the origin. Show how a power series approximation to the function h may be computed, and compute the first two non-zero terms of the power series of the unstable manifold. 8

4. (a) Define stability and asymptotic stability in the sense of *Lyapunov* for the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

2

(b) Describe *Lyapunov*'s 2nd or Direct Method, clearly stating the relevant theorems. 4

(c) State *La Salle*'s Invariance Principle and discuss how it may be used to determine asymptotic stability. 4

(d) Determine whether the origin of

$$\dot{x} = y, \qquad \dot{y} = -x|x| - \sinh y$$

is an asymptotically stable fixed point. Is it globally asymptotically stable? 10

5. (a) State the *Poincaré-Bendixson* theorem. 4

(b) Use the theorem to show that

$$\dot{x} = x(2x + 8 - x^2 - y^2) - y, \qquad \dot{y} = y(2x + 8 - x^2 - y^2) + x$$

has a closed orbit. 8

(c) Using the positive x-axis as a surface of section, without doing any calculations show how the *Poincaré* map associated with the flow of part (b) may be computed, and the stability of its fixed point determined. 8

6. The equation

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - \frac{BN}{A + N}$$

where $R,\ K,\ A$ and $B > 0$ is a model of the population growth dynamics of a species of moth; in isolation the species would grow logistically but in its usual habitat it is subject to predation by birds. $N(t)$ is the population density at time t, R is the intrinsic growth rate, K is the carrying capacity and A and B are parameters that set the level of predation.

(a) Show that the system can be rescaled as

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x}{1+x}$$

for suitably defined dimensionless quantities $x,\ \tau,\ r$ and k. 6

(b) With $k = 4$, find the fixed points of the system of part(a) as a function of the parameter r. 4

(c) Do bifurcations occur as r varies? If so, classify them and find the corresponding critical values of r. 8

(d) Give a ecological interpretation of how the model behaves. 2

7. (a) State the *Neimark-Sacker* bifurcation theorem for a two-dimensional map. 4

(b) Let

$$x' = x - 2y, \qquad y' = r - x^2$$

where $r > 0$.
Find the fixed points of the system as functions of the parameter r and classify them. 8

(c) For the system of part (b), use the *Neimark-Sacker* theorem to show that one of the fixed points undergoes a *Neimark-Sacker* bifurcation at a certain value of the parameter $r = r_c$. Find the value r_c. 8