UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018

SEMESTER: Spring 2012

MODULE TITLE: Dynamical Systems

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 100 %

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time vector sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

 Marks

1. The following dynamical system represents interaction between two self-limiting populations of normalised sizes $x(t), y(t) \geq 0$:

$$\dot{x} = 2x\left(1 - \frac{3x}{3+y}\right), \qquad \dot{y} = 2y\left(1 - \frac{y}{3+x}\right)$$

   (a) Find the fixed points of the system. 4

   (b) Use linearisation to characterise the nature of the fixed points. 8

   (c) Show that $y = 0$ and $x = 0$ are trajectories of the system. 2

   (d) Can the system exhibit closed orbits ? Explain your answer. 4

   (e) Is the interaction competitive, mutualistic or of a predator-prey nature? Explain your answer. 2

2. Consider the 2-parameter dynamical system which represents an overlapping generations model for a single species

$$x' = rxe^{-x} + \mu x$$

   where x is the population size in the current time period, $r > 0$ is the growth factor associated with new births appearing in the next generation and $0 \leq \mu \leq 1$ is the proportion of the current generation which survives to the next time period.
   With $\mu = 0.5$

   (a) find the fixed points of the map, and determine their stability properties as a function of the parameter r. 8

   (b) Classify any bifurcations that occur. 8

   (c) State *Sharkovsky*'s theorem. It can be shown that there are no period-4 points for $r < R_4 \approx 90$. What can be said about other periodic orbits for $r < R_4$? 4

3. (a) Define stability and asymptotic stability in the sense of *Lyapunov* for the dynamical system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   3

   (b) Describe *Lyapunov*'s 2nd or Direct Method, clearly stating the relevant theorems. 5

(c) Show that the origin of

$$x' = y, \qquad y' = -\frac{1}{4}x - y$$

is a globally asymptotically stable fixed point by constructing a quadratic function which is a strict *Lyapunov* function. 8

(d) Will the function calculated in part(c) act as a *Lyapunov* function for the origin of the system

$$x' = y, \qquad y' = -\frac{1}{4}x - y + x^2.$$

Explain your answer. 4

4. (a) Define the centre manifold of a fixed point of a flow. 4

(b) Show that the system

$$\dot{x} = xy, \qquad \dot{y} = -2y + x^2$$

has a non-hyperbolic fixed point at the origin. In a neighbourhood of the origin, the centre manifold can be written as $y = h(x)$. Show how a power series approximation to the function h may be computed, and compute the first two non-zero terms of h. 8

(c) Use the centre manifold calculated in part (b) to investigate the local stability properties of the system near the origin. 8

5. (a) State the *Poincaré-Bendixson* theorem. 4

(b) Use the theorem to show that the modified *Sel'kov* model of glycolysis given by

$$\dot{x} = -16x + y + x^2y, \qquad \dot{y} = 20 - y - x^2y$$

has a closed orbit. 16

6. (a) State the *Andronov-Hopf* bifurcation theorem. 4

(b) Use the theorem to show that the fixed point of the *Brusselator*, a theoretical model of an autocatalytic reaction exemplified by

$$\dot{x} = 1 + x^2y - (b+1)x, \qquad \dot{y} = bx - x^2y$$

undergoes a *Hopf* bifurcation at a value of the parameter $b = b_c$. Find the value b_c. 12

(c) Find the approximate period of the limit cycle for $b \approx b_c$. 4

7. (a) Describe the *Ott-Grebogi-Yorke* (OGY) control method as it might be applied to the dynamical system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}; p)$$

with parameter p to stabilise the unstable fixed point $\mathbf{x}_e$ (corresponding to the nominal value $p = p_0$) which is embedded in a chaotic attractor. 10

(b) What advantages and disadvantages does the method have? 5

(c) The map

$$x' = pxe^{-x}$$

is known to be chaotic for $p = p_0 = 20$. If the system is being driven at this nominal parameter value, construct the OGY controller that stabilises the non trivial fixed point x_e making it superstable when "active": $|x - x_e| < \epsilon$. Find the value for ϵ that corresponds to an allowable parameter variation of $\pm 1\%$. 5