UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018

SEMESTER: Spring 2014

MODULE TITLE: Dynamical Systems

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 90 %

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time vector sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

1. The following dynamical system represents the interaction between a prey population and a predator population of sizes $x(t),\ y(t) \geq 0$ respectively in normalised units.

$$\dot{x} = \frac{3}{2}x\left(1 - \frac{x}{60}\right) - y\left(\frac{x}{50+x}\right), \qquad \dot{y} = y\left(-1 + 2\left(\frac{x}{50+x}\right)\right).$$

   (a) Find the fixed points of the system. 3

   (b) Use linearisation to characterise the nature of the fixed points. 8

   (c) Show that $y = 0$ and $x = 0$ are trajectories of the system. 2

   (d) Can the system exhibit closed orbits? Explain your answer. 5

2. In a duopoly two firms dominate a market and their interactions determine how much each firm produces and how much a product sells for. In a dynamic model of duopoly, at every decision point each firm produces a number of items designed to maximise its profits in reaction to what the other firm previously produced. The regularised *Cournot-Puu* duopoly model is

$$X' = \sqrt{\frac{Y}{a}} - Y, \qquad Y' = \sqrt{\frac{X}{b}} - X$$

where $X \geq 0$ and $Y \geq 0$ are the numbers of items previously produced by the firms with marginal costs $a > 0$ and $b > 0$ respectively, and X' and Y' are the numbers to be produced at this decision point. It is known that the model is consistent (i.e. X and Y remain positive if the initial values satisfy $bX \leq 1$ and $aY \leq 1$) whenever

$$\frac{1}{6.25} \leq r \leq 6.25$$

where $r = a/b$ is the ratio of the marginal costs.

   (a) By choosing an appropriate rescaling of the variables show that the map can be rewritten as

$$x' = \frac{1}{r}\left(\sqrt{y} - y\right), \qquad y' = r\left(\sqrt{x} - x\right).$$

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   (b) Find the fixed points of the map and investigate their stability as a function of the parameter r. 10

   (c) What happens as r goes through the value $3 + 2\sqrt{2}$? 4

3. (a) For the invertible discrete dynamical system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   where the origin is a saddle point, define the stable and unstable manifolds. 2

(b) Compute the stable and unstable manifolds of the linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \mathbf{x}.$$

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(c) The system

$$x' = x + \frac{1}{2}y + xy^2, \qquad y' = \frac{1}{2}x + y + x^2 y$$

with a saddle point at the origin, has an invariant manifold which may be written as

$$y = h(x) = x.$$

Is it the stable or unstable manifold? 4
Compute the other manifold. 6

4. (a) Define stability and asymptotic stability in the sense of *Lyapunov* for the dynamical system 2

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}).$$

(b) Describe *Lyapunov*'s 2nd or Direct Method, clearly stating the relevant theorems. 2

(c) Determine a value of β for which $V(x, y) = x^2 + 2xy + \beta y^2$ is a strict global *Lyapunov* function for the origin of the system 5

$$\dot{x} = -2x + y + xy + 3y^2, \qquad \dot{y} = x - 2y - xy - y^2.$$

(d) State *La Salle*'s Invariance Principle and discuss how it may be used to determine asymptotic stability. 4

(e) Using the candidate *Lyapunov* function $V(x, y) = x^6 + y^4$, show that the origin of

$$\dot{x} = -x + y^3, \qquad \dot{y} = -\frac{3}{2}x^5$$

is asymptotically stable. Is it globally asymptotically stable? 5

5. The modified *Schnakenberg* model of a chemical oscillator is

$$\dot{x} = x^2 y - x + b, \qquad \dot{y} = -x^2 y + a.$$

(a) For $a = 0.32,\ b = 0.18$, find the fixed point of the system and classify it. 4

(b) Draw the nullclines for the system above, and plot typical flow directions along these nullclines and along the boundary of the quadrilateral with vertices $(b, 0),\ (b, a/b^2),\ (1, 1 - b)$ and $(1, 0)$. 8

(c) State the *Poincaré-Bendixson* theorem, and use it to prove that the *Schnackenberg* model has a closed orbit. 6

6. (a) State the *Andronov-Hopf* bifurcation theorem. 2

   (b) The *Kaldor* business cycle model is

$$\dot{Y} = \alpha(I - S), \qquad \dot{K} = I - \delta K$$

   where Y is output or production, I investment, S savings, K capital, α a speed-of-adjustment factor for excess demand and δ the capital depreciation rate. In addition,

$$I = \sigma(Y) - \beta K, \qquad S = \gamma Y$$

   where $\sigma(Y)$ is a sigmoidal function of the output, β is termed a costs-adjustment coefficient and γ represents the propensity to save. For the parameter values $\delta = 0.05$, $\beta = 0.55$ and $\gamma = 0.1$, and $\sigma(Y)$ given by

$$\sigma(Y) = \begin{cases} 0.7, & \text{if } Y \leq 0.5 \\ Y + 0.2, & \text{if } 0.5 < Y < 1.5 \\ 1.7, & \text{if } 1.5 \leq Y \end{cases}$$

   show that the system has a fixed point at

$$Y_e = 1, \qquad K_e = 2$$

   for all values of α. 2

   (c) Show that the fixed point found in part (b) undergoes a *Hopf* bifurcation at a value $\alpha = \alpha_c$ and find α_c. 8

   (d) Describe the behaviour of the model for $\alpha < \alpha_c$ and for $\alpha > \alpha_c$. 2

   (e) Find the approximate period of the limit cycle for $\alpha \approx \alpha_c$. 2

   (f) Define what is (i) a supercritical *Hopf* bifurcation and (ii) a subcritical *Hopf* bifurcation. If you are told that the limit cycle born at the bifurcation is stable, is the *Hopf* bifurcation of part(c) super- or subcritical? Explain you answer. 2

7. A fishery with both juvenile and adult fish in which only adult fish are harvested is modelled by

$$J_{k+1} = rA_k, \qquad A_{k+1} = fJ_k + sA_k - hA_k$$

   where J_k and A_k are the juvenile and adult population densities in season k respectively, r is the birth rate of juveniles per head of adult population, f and s are the survival rates of juveniles and adults respectively and h is the proportion of adults harvested per season. It is wished to investigate the effect of choosing h on the sustainability of the fishery. It is known that $r = 2$, $f = 0.6$ but s is determined by competition for resources as $s = 1 - 1.5J_k - 0.5A_k$.

   (a) With an appropriate rescaling $J = mx$ and $A = ny$ show that the model can be written as

$$x' = y, \qquad y' = 1.2x + (1 - 6x - y)y - hy.$$

   4

(b) In the absence of harvesting ($h = 0$) determine the fixed points of the rescaled system and their stability properties. Is the model consistent (i.e. all appropriate values remain nonnegative)for this case? 4

(c) For what range of values of h is the model consistent and the positive fixed point stable. What is the maximum number of adults that can be harvested when the system is in equilibrium? 8

(d) What happens if h is increased so that the positive fixed point is destabilised? Is this scenario sustainable? 2