Let the proper divisors of p be $p_1 = 1, p_2, \ldots, p_m$. The equation $f^{(p)}(x) = x$ has as solutions all points which return to themselves after p iterations under the map f. Hence the 'prime' (or true) period-p points are solutions of the above equation with the lower period points factored out, i.e. solutions of

$$\frac{f^{(p)}(x) - x}{(f(x)-x)(f^{(p_2)}(x)-x)\cdots(f^{(p_m)}(x)-x)} = 0$$

Furthermore if $x_1, x_2, \ldots, x_p$ are the points of a period-p orbit, then stability is checked by computing the absolute value of the 'multiplier' λ, i.e. $|\lambda| = |f'(x_p)f'(x_{p-1})\cdots f'(x_2)f'(x_1)|$

1. When $f(x) = 1 - x^2$, $f'(x) = -2x$ and

$$\begin{aligned}\frac{f^{(2)}(x)-x}{f(x)-x} = \frac{1-(1-x^2)^2-x}{1-x^2-x} &= x(1-x)\\ &= 0 \Rightarrow x = 0, 1\end{aligned}$$

The associated multiplier is $|-2(0) \times -2(1)| = 0$. Hence the orbit is superstable.

When $f(x) = 2x$ and

$$\begin{aligned}\frac{f^{(2)}(x)-x}{f(x)-x} = \frac{2(2x)-x}{2x-x} &= 3\\ &= 0 \Rightarrow \text{no solution}\end{aligned}$$

Hence there are no period-2 orbits.

When $f(x) = rx(1-x)$, $f'(x) = r(1-2x)$ and

$$\begin{aligned}\frac{f^{(2)}(x)-x}{f(x)-x} = \frac{r^2x(1-x)(1-rx(1-x))-x}{rx(1-x)-x} &= r^2x^2 - (r^2+r)x + r + 1\\ &= 0 \Rightarrow x = \frac{r+1 \pm \sqrt{(r+1)(r-3)}}{2r}\end{aligned}$$

Thus there exists a (real) 2-orbit only when $r > 3$ (or $r < -1$ which we ignore). Its multiplier is

$$\left| r\left(1 - 2\frac{r+1-\sqrt{(r+1)(r-3)}}{2r}\right) \times r\left(1 - 2\frac{r+1+\sqrt{(r+1)(r-3)}}{2r}\right)\right| = |r^2 - 2r - 4|$$

and it is stable when $3 < r < 1 + \sqrt{6} \approx 3.4495$.

When $f(x) = 10xe^{-x}$, $f'(x) = 10(1-x)e^{-x}$ and

$$\begin{aligned}\frac{f^{(2)}(x)-x}{f(x)-x} &= \frac{10(10xe^{-x})e^{-10xe^{-x}} - x}{10xe^{-x} - x}\\ &= 0 \Rightarrow x = 0.9345958231,\ \ 3.670574363\ \ (\ \text{numerically})\end{aligned}$$

The associated multiplier is $|10(1-0.935)e^{-0.935} \times 10(1-3.671)e^{-3.671}| = 0.1746667177$. Hence the orbit is stable.

2.

$$f^{(3)}(x) = \begin{cases} 8x, & \text{if } 0 < x \le 1/8 \\ 2 - 8x, & \text{if } 1/8 < x \le 1/4 \\ 8x - 2, & \text{if } 1/4 < x \le 3/8 \\ 4 - 8x, & \text{if } 3/8 < x \le 1/2 \\ 8x - 4, & \text{if } 1/2 < x \le 5/8 \\ 6 - 8x, & \text{if } 5/8 < x \le 3/4 \\ 8x - 6, & \text{if } 3/4 < x \le 7/8 \\ 8 - 8x, & \text{if } 7/8 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f'(x) = \begin{cases} 2, & \text{if } 0 < x < 1/2 \\ -2, & \text{if } 1/2 < x < 1 \\ \text{not defined}, & \text{otherwise} \end{cases}$$

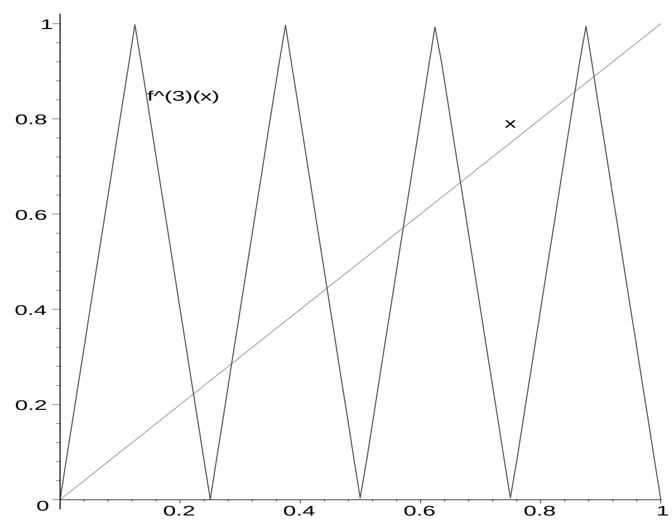

Figure 1: $f^{(3)}(x)$ and x v. x

The solutions of $f^{(3)}(x) = x$ are $0,\ 2/9,\ 2/7,\ 4/9,\ 4/7,\ 2/3,\ 6/7,\ 8/9$ of which 0 and 2/3 are fixed points, while 2/9, 4/9 and 8/9 form one 3-cycle with multiplier $|2 \times 2 \times -2| = 8$ while 2/7, 4/7 and 6/7 form a second 3-cycle with multiplier $|2 \times -2 \times -2| = 8$. Hence both cycles are unstable.

3. $f(x) = 4x(1-x)$, $f'(x) = 4(1-2x)$ and

$$\begin{aligned} \frac{f^{(3)}(x) - x}{f(x) - x} &= 4096x^6 - 13312x^5 + 16640x^4 - 10048x^3 + 3024x^2 - 420x + 21 \text{ using Maple} \\ &= 0 \Rightarrow x = 0.1169777784,\ 0.1882550991,\ 0.4131759112, \\ &\quad 0.6112604670,\ 0.9504844340,\ 0.9698463104 \text{ numerically} \end{aligned}$$

of which 0.1170, 0.4132 and 0.9698 form a 3-cycle with multiplier $|3.0642 \times 0.6946 \times -3.7588| = 8$, while 0.1883, 0.6113 and 0.9698 form a second 3-cycle with multiplier $|2.4940 \times -0.8901 \times -3.6039| = 8$. Hence both cycles are unstable.

4. One approach is to use Maple and play around with r values until a stable 3-period is found ☺. It is known that period-3 orbits are stable for $1+\sqrt{8} \approx 3.8284 < r < 3.8414$.