Let $\chi(\lambda)$ represent the characteristic polynomial of a matrix.

1. (a) $\mathbf{x}_e = \mathbf{0}$ is a stable focus since:

$$A = \begin{pmatrix} 2 & -3 \\ 3 & -3 \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2+\lambda+3 = (\lambda-1/2+i\sqrt{13}/2)(\lambda-1/2-i\sqrt{13}/2)$$

(b) $\mathbf{x}_e = \mathbf{0}$ is a saddle since:

$$A = \begin{pmatrix} 1 & -2 \\ 4 & -5 \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)$$

(c) $\mathbf{x}_e = \mathbf{0}$ is an unstable node since:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2 - 9\lambda + 14 = (\lambda-2)(\lambda-7)$$

$\mathbf{x}_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a stable node since:

$$A = \begin{pmatrix} -1 & -6 \\ 0 & -7/2 \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2 + (9/2)\lambda + 7/2 = (\lambda+1)(\lambda+7/2)$$

$\mathbf{x}_e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a stable node since:

$$A = \begin{pmatrix} -1 & 0 \\ -21/5 & -7 \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2 + 8\lambda + 7 = (\lambda+1)(\lambda+7)$$

$\mathbf{x}_e = \begin{pmatrix} 4/7 \\ 1/7 \end{pmatrix}$ is a saddle since:

$$A = \begin{pmatrix} -8/7 & -24/7 \\ -3/2 & -1 \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2+(15/7)\lambda-4 = (\lambda+\frac{15+\sqrt{799}}{14})(\lambda+\frac{15-\sqrt{799}}{14})$$

2. (i) The ideas follow more or less directly those of flows, with the difference that stability is determined by the moduli of the eigenvalues. Hence we have

Eigenvalues	Classification of $\mathbf{x}_e = \mathbf{0}$
Real, $\lvert\lambda_1\rvert,\ \lvert\lambda_2\rvert < 1$	stable node (sink)
Real, $\lvert\lambda_1\rvert,\ \lvert\lambda_2\rvert > 1$	unstable node (source)
Real, $\lvert\lambda_1\rvert < 1,\ \lvert\lambda_2\rvert > 1$	saddle
Complex, $\lvert\lambda_1\rvert,\ \lvert\lambda_2\rvert < 1$	stable focus (sink)
Complex, $\lvert\lambda_1\rvert,\ \lvert\lambda_2\rvert > 1$	unstable focus (source)
Complex, $\lvert\lambda_1\rvert = 1 = \lvert\lambda_2\rvert$	centre

The last entry corresponds to a non-hyperbolic case. What about the other non-hyperbolic cases ? (Traditionally, they are referred to as degenerate cases.)

(ii) $\chi(\lambda) = \lambda^2 - a\lambda + b = (\lambda-\lambda_1)(\lambda-\lambda_2) = 0$, where $a = \lambda_1+\lambda_2,\ b = \lambda_1\lambda_2$, has solutions

$$\lambda_{1,2} = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

We'll investigate the classification under the following headings.

(a) Real Distinct Eigenvalues: $\left(\frac{a}{2}\right)^2 > b$.
In the case of a sink (or stable node in this situation: $|\lambda_1|, |\lambda_2| < 1 \quad\Rightarrow\quad |b| < 1$)

$$\begin{aligned} -1 &< \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} < 1 \\ -1 &< \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} < 1 \end{aligned}$$

from which it follows that

$$-1 < \frac{a}{2} < 1 \qquad \Rightarrow \qquad \frac{|a|}{2} < 1 \tag{1}$$

and also

$$\begin{aligned} \frac{a}{2} - 1 &< \sqrt{\left(\frac{a}{2}\right)^2 - b} < \frac{a}{2} + 1 \\ -\frac{a}{2} - 1 &< \sqrt{\left(\frac{a}{2}\right)^2 - b} < -\frac{a}{2} + 1 \end{aligned}$$

which using equation (1) yields

$$\sqrt{\left(\frac{a}{2}\right)^2 - b} < 1 - \frac{|a|}{2} \qquad \Rightarrow \qquad |a| < 1 + b = |1 + b| \tag{2}$$

While in the case of a source (or unstable node here: $|\lambda_1|,\ |\lambda_2| > 1$
$\Rightarrow \quad |b| > 1 \quad \Rightarrow \quad |a| > 2$)
Either

$$\begin{aligned} 1 < \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} &< \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} \\ \Rightarrow \sqrt{\left(\frac{a}{2}\right)^2 - b} &< \frac{a}{2} - 1 \\ \Rightarrow 2 < a &< 1 + b \end{aligned} \tag{3}$$

or

$$\begin{aligned} \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} &< \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} < -1 \\ \Rightarrow \sqrt{\left(\frac{a}{2}\right)^2 - b} &< -\frac{a}{2} - 1 \\ \Rightarrow -(1 + b) &< a < -2 \end{aligned} \tag{4}$$

or

$$\begin{aligned} \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} < -1 &< 1 < \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} \\ \Rightarrow 1 + \frac{|a|}{2} &< \sqrt{\left(\frac{a}{2}\right)^2 - b} \\ \Rightarrow |a| &< -1 - b \end{aligned} \tag{5}$$

Equations (3),(4) and (5) can be summarised by

$$|a| < |1+b| \tag{6}$$

Finally in the case of a saddle ($|\lambda_1| < 1,\ |\lambda_2| > 1$)
Either

$$\begin{aligned} -1 < \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} &< 1 < \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} \\ \Rightarrow |1 - \frac{a}{2}| &< \sqrt{\left(\frac{a}{2}\right)^2 - b} < 1 + \frac{a}{2} \\ \Rightarrow |1+b| &< a \end{aligned} \tag{7}$$

or

$$\begin{aligned} \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} < -1 &< \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} < 1 \\ \Rightarrow |1 + \frac{a}{2}| &< \sqrt{\left(\frac{a}{2}\right)^2 - b} < 1 - \frac{a}{2} \\ \Rightarrow |1+b| &< -a \end{aligned} \tag{8}$$

From equations (7) and (8), we conclude that

$$|a| > |1+b| \tag{9}$$

(b) Complex Eigenvalues: $\left(\frac{a}{2}\right)^2 < b$.
It's straightforward to calculate that $|\lambda| = |b|$ and thus it follows that $|b| < 1$ gives a stable focus, $|b| > 1$ an unstable focus and $|b| = 1$ a centre.

(c) Real Equal Eigenvalues: $\left(\frac{a}{2}\right)^2 = b$.
Now $|\lambda| = |a/2|$ and so $|a| < 2$ corresponds to a stable node, $|a| > 2$ an unstable node, while $|a| = 2$ is a degenerate case which requires further knowledge of the eigenstructure of the Jacobian of the system in order to classify.

Summarising (see Fig. 1 also)

Classification of $\mathbf{x}_e = \mathbf{0}$	Relationship between a and b
stable node (sink)	$\|a\| < \|1+b\|,\ \|b\| < 1,\ (a/2)^2 > b$
unstable node (source)	$\|a\| < \|1+b\|,\ \|b\| > 1,\ (a/2)^2 > b$
saddle	$\|a\| > \|1+b\|$
stable focus (sink)	$\|a\| < \|1+b\|,\ \|b\| < 1,\ (a/2)^2 < b$
unstable focus (source)	$\|a\| < \|1+b\|,\ \|b\| > 1,\ (a/2)^2 < b$
centre	$\|a\| < 2,\ b = 1$

Note: $|a| = |1+b|$ corresponds to the non-hyperbolic (degenerate) cases $|\lambda_1| = 1,\ |\lambda_2| = |b|$.

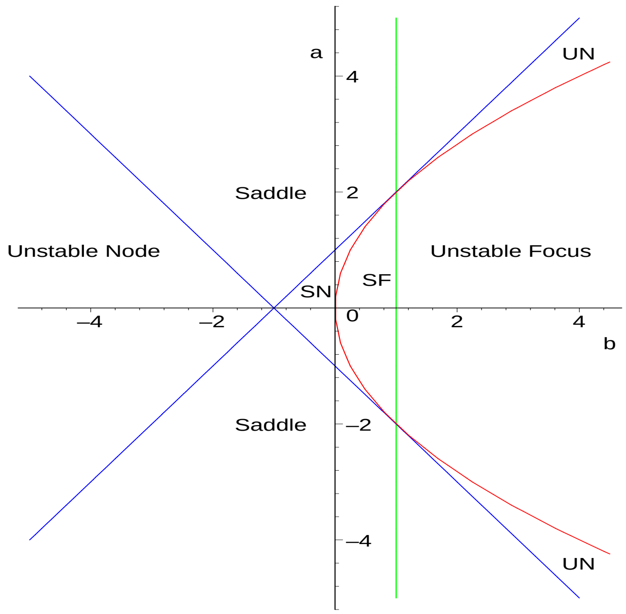

Figure 1: Stability in $a-b$ plane: SN = stable node, UN = unstable node, SF = stable focus

3. (a) $\mathbf{x}_e = \mathbf{0}$. $\chi(\lambda) = \lambda^2 - 1.5\lambda + 0.6$. Stable focus.
   (b) $\mathbf{x}_e = \mathbf{0}$. $\chi(\lambda) = \lambda^2 - 1.75\lambda + 0.625$. Saddle.
   (c) (i) $\mathbf{x}_e = \mathbf{0}$. $\chi(\lambda) = \lambda^2 - 0.4$. Stable node.
   (ii) $\mathbf{x}_e = (-0.6,\ -0.6)^T$. $\chi(\lambda) = \lambda^2 - 1.2\lambda - 0.4$. Saddle.

4. Yes.
   The period-p points of the map $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ are fixed points of the map $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, where

$$\mathbf{F} = \underbrace{\mathbf{f} \circ \mathbf{f} \circ \cdots \circ \mathbf{f}}_{p \text{ times}}$$

   The fixed points of $\mathbf{F}$ are classified by the eigenvalues of its Jacobian $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}$ in the same fashion as that for $\mathbf{f}$ either as saddles, nodes, foci etc. Recall that for the periodic orbit $\mathbf{o_p} = \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_p) \cdots \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_2) \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_1)$$

5. (a)

$$\begin{pmatrix} 0.7 \\ -0.1 \end{pmatrix} \mapsto \begin{pmatrix} 0.43 - (0.7)^2 + 0.4(-0.1) \\ 0.7 \end{pmatrix} = \begin{pmatrix} -0.1 \\ 0.7 \end{pmatrix} \mapsto \begin{pmatrix} 0.43 - (-0.1)^2 + 0.4(0.7) \\ -0.1 \end{pmatrix} = \begin{pmatrix} 0.7 \\ -0.1 \end{pmatrix}$$

   (b) The first two iterations of the map yield

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} p - x^2 + qy \\ x \end{pmatrix} \mapsto \begin{pmatrix} p - (p - x^2 + qy)^2 + qx \\ p - x^2 + qy \end{pmatrix}$$

   Thus the fixed point(s) satisfy

$$x = p - x^2 + qy, \qquad y = x, \qquad \Rightarrow \qquad x^2 + (1-q)x - p = 0 \qquad (10)$$

and the period-2 points satisfy

$$
\begin{aligned}
& x = p - (p - x^2 + qy)^2 + qx, \qquad\qquad y = p - x^2 + qy, \\
\Rightarrow \qquad & y = \frac{p - x^2}{1 - q}, \qquad\qquad x^4 - 2px^2 + (1-q)^3 x + p^2 - (1-q)^2 p = 0 \qquad (11)
\end{aligned}
$$

Therefore the x-coordinates of the (prime) period-2 points are given by the roots of

$$
\begin{aligned}
\frac{x^4 - 2px^2 + (1-q)^3 x + p^2 - (1-q)^2 p}{x^2 + (1-q)x - p} &= x^2 - (1-q)x + (1-q)^2 - p = 0 \\
\Rightarrow x_{1,2} &= \frac{1-q}{2} \pm \sqrt{p - 3\left(\frac{1-q}{2}\right)^2} \qquad (12)
\end{aligned}
$$

and thus (for q=0.4), there are real period-2 points only when

$$
p > 3\left(\frac{1-0.4}{2}\right)^2 = 0.27
$$

and the period-2 points are $0.3 \pm \sqrt{p - 0.27}$.

The Jacobian of the Hénon map is

$$
\begin{pmatrix} -2x & q \\ 1 & 0 \end{pmatrix}
$$

Thus the "Jacobian " of the 2-cycle map is

$$
\begin{pmatrix} -2(0.3 + \sqrt{p - 0.27}) & 0.4 \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} -2(0.3 - \sqrt{p - 0.27}) & 0.4 \\ 1 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1.84 - 4p & -0.24 - 0.8\sqrt{p - 0.27}) \\ -0.6 + 2\sqrt{p - 0.27} & 0.4 \end{pmatrix}
$$

which has $\chi(\lambda) = \lambda^2 - (2.24 - 4p)\lambda + 0.16$. Thus the 2-cycle is stable (why ?) if

$$
|2.24 - 4p| < 1.16 \qquad \Leftrightarrow \qquad 0.27 < p < 0.85
$$