1. $\mathbf{x}_e = \mathbf{0}$ is a stable node since the Jacobian is

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \chi(\lambda) = (\lambda+1)^2$$

$\mathbf{x}_e = \begin{pmatrix} -8/3 \\ 8 \end{pmatrix}$ is a saddle since the Jacobian is

$$A = \begin{pmatrix} 3 & -1/3 \\ -6 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \chi(\lambda) = \lambda^2 - 2\lambda - 5 = (\lambda - 1 + \sqrt{6})(\lambda - 1 - \sqrt{6})$$

$\mathbf{x}_e = \begin{pmatrix} 2/3 \\ 1/2 \end{pmatrix}$ is a saddle since the Jacobian is

$$A = \begin{pmatrix} -3/4 & 4/3 \\ 3/2 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \chi(\lambda) = \lambda^2 + 7/4\lambda - 5/4 = (\lambda + (7+\sqrt{129})/8)(\lambda + (7-\sqrt{129})/28)$$

(a)

$$\begin{aligned} V(x,y) &= 2x^2 + 2xy + 3y^2 \\ \Rightarrow \dot{V} &= 4x\dot{x} + 2\dot{x}y + 2x\dot{y} + 6y\dot{y} \\ &= (4x+2y)(-x+y+\frac{1}{2}xy) + (2x+6y)(-y+\frac{9}{8}x^2) \\ &= -4x^2 - 4y^2 + (xy^2 + \frac{35}{4}x^2y + \frac{9}{4}x^3) \qquad (1) \end{aligned}$$

The bracketed terms at the right of Equation(2) are of third order, and hence will be negligible in comparison to the quadratic terms for small $\mathbf{x}$. Thus $\dot{V} < 0$ for $\mathbf{x}$ near $\mathbf{0}$, and since V is positive definite (why ?), it is a *Lyapunov* function. To quantify this and estimate the basin of attraction for $\mathbf{0}$, consider the following minimisation problem:

$$\min V \qquad \text{subject to } \dot{V} = 0, \ \mathbf{x} \neq \mathbf{0}$$

This has solution $V = V^* = 1.981706687$ (numerically). Thus the estimate of the basis of attraction (see Fig. 1) is

$$2x^2 + 2xy + 3y^2 < V^*$$

(b) Only done for one of the saddles: $(2/3, 1/2)^T$. (See Fig. 1).

(c) From Fig.1, the estimate of the basin of attraction cannot be substantially improved using an ellipsoidal region since the basin of attraction must be bounded by $W^s((2/3, 1/2)^T)$.

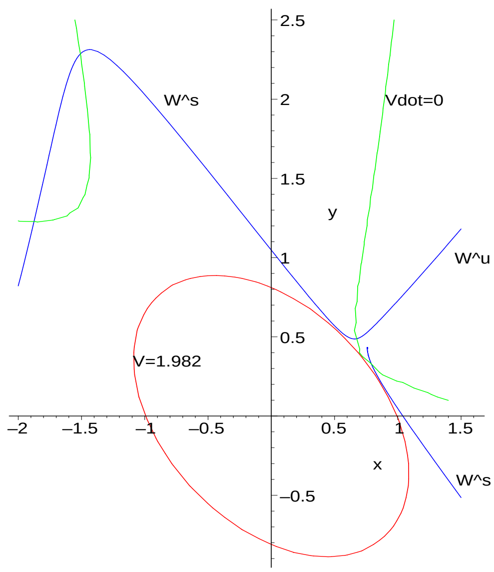

Figure 1: Q3: Estimate of Basin of Attraction for $\mathbf{0}$. Part of the invariant manifolds for $(2/3, 1/2)^T$

2.

$$
\begin{aligned}
V(x,y) &= ax^2 + bxy + cy^2 \\
\Rightarrow \dot{V} &= 2ax\dot{x} + b\dot{x}y + bx\dot{y} + 2cy\dot{y} \\
&= (2ax + by)(y - x^3) + (bx + 2cy)(-x - x^3) \\
&= -2ax^4 + (2a - 2c)xy - 2cy^4 + b(y^2 - x^3y - x^2 - xy^3) \qquad (2)
\end{aligned}
$$

Choose $c = a > 0$ and $b = 0$. Equation(3) becomes

$$V = a(x^2 + y^2) \qquad \Rightarrow \qquad \dot{V} = -2a(x^4 + y^4)$$

so V is a global strict *Lyapunov* function.

3. If V is a global strict *Lyapunov* function for $\mathbf{x}_e$, then all orbits are attracted (asymptotically) to $\mathbf{x}_e$, and hence there cannot be any periodic orbits.

4. The *Lyapunov* equation is $A^TP + PA = -Q$. We'll use $Q = I$ in each of the following:

   (a)

$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} -2p_2 &= -1 \\ p_1 + 2p_2 - p_3 &= 0 \\ 2p_2 + 4p_3 &= -1 \end{aligned}$$

which gives

$$P = \begin{pmatrix} -3/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

which is not positive definite (principal minors are -3/2 and 1/2), and so $\mathbf{x}_e = \mathbf{0}$ is not asymptotically stable. In fact P is negative definite, and so invoking Theorem 3 of *Lyapunov*'s direct method, since $V(\mathbf{x})$ and $\dot{V}$ have the same sign, $\mathbf{x}_e = \mathbf{0}$ is unstable.

(b)

$$\begin{pmatrix} 0 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} -2p_2 &= -1 \\ p_1 - 4p_2 - p_3 &= 0 \\ 2p_2 - 8p_3 &= -1 \end{aligned}$$

which gives

$$P = \begin{pmatrix} 9/4 & 1/2 \\ 1/2 & 1/4 \end{pmatrix}$$

which is positive definite (principal minors are 9/4 and 5/16), and so $\mathbf{x}_e = \mathbf{0}$ is asymptotically stable.

(c)

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} -2p_3 &= -1 \\ p_1 - 3p_3 - p_5 &= 0 \\ p_2 - 3p_3 - p_6 &= 0 \\ 2p_2 - 6p_5 &= -1 \\ p_3 + p_4 - 3p_5 - 3p_6 &= 0 \\ 2p_5 - 6p_6 &= -1 \end{aligned}$$

which gives

$$P = \begin{pmatrix} 37/16 & 31/16 & 1/2 \\ 31/16 & 13/4 & 13/16 \\ 1/2 & 13/16 & 7/16 \end{pmatrix}$$

which is positive definite (principal minors are 37/16, 963/256 and 457/512), and so $\mathbf{x}_e = \mathbf{0}$ is asymptotically stable.

5. When the quadratic function $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ is used for the linear map $\mathbf{x}' = A\mathbf{x}$, we get

$$\triangle V = V(\mathbf{x}') - V(\mathbf{x}) = \mathbf{x}^T A^T PA\mathbf{x} - \mathbf{x}^T P\mathbf{x} = \mathbf{x}^T(A^T PA - P)\mathbf{x}$$

Defining $Q = -(A^T PA - P)$, and noting that $Q^T = Q$, we have the following theorem:
For positive definite Q, the solution P of the discrete *Lyapunov* equation

$$A^T PA - P = -Q \qquad (3)$$

is positive definite if and only if A is a "convergent" matrix (defined as a matrix all of whose eigenvalues have modulus less than 1).
For the given system, with $Q = I$ Equation (4) becomes

$$\begin{pmatrix} 0 & 0.5 \\ 1 & 0.5 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} - \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} 0.25p_3 - p_1 &= -1 \\ -0.5p_2 + 0.25p_3 &= 0 \\ p_1 + p_2 - 0.75p_3 &= -1 \end{aligned}$$

which gives no solution for P. Hence there is no positive definite solution for P and so $\mathbf{x}_e = \mathbf{0}$ is not asymptotically stable. (Check that the eigenvalues of A are -1/2 and 1, which means what ?)