1. The *Jacobian* matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -b \end{bmatrix}$$

(a) (i)

$$\mathbf{x}_e = \mathbf{0}, \qquad D\mathbf{f}(\mathbf{0}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \chi(\lambda) &= (\lambda + b)(\lambda^2 + (\sigma+1)\lambda + \sigma(1-\rho)) \\ &= (\lambda + 8/3)\left(\lambda^2 + 11\lambda + 10(1-\rho)\right). \end{aligned} \tag{1}$$

The three eigenvalues are $-8/3, (-11 \pm \sqrt{81+40\rho})/2$. For $\rho < 1$ this classifies the origin as a stable node; for $\rho > 1$ and in particular for $\rho = 28$ it is a saddle.

(ii) Letting $v = \sqrt{b(\rho-1)} \approx 8.485$, the other fixed points are

$$\mathbf{x}_e = \begin{pmatrix} \pm v \\ \pm v \\ \rho - 1 \end{pmatrix}, \qquad D\mathbf{f}(\mathbf{x}_e) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp v \\ \pm v & \pm v & -b \end{bmatrix}.$$

Thus these fixed points only exist if $\rho > 1$. In fact, a pitchfork bifurcation occurs at $\rho = 1$.

The characteristic polynomial (of each fixed point) is

$$\begin{aligned} \chi(\lambda) &= \lambda^3 + (\sigma + b + 1)\lambda^2 + (b + v^2 + \sigma b)\lambda + 2\sigma v^2 \\ &= \lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + \rho)\lambda + 2\sigma b(\rho - 1) \\ &= \lambda^3 + \frac{41}{3}\lambda^2 + \frac{8}{3}(10 + \rho)\lambda + \frac{160}{3}(\rho - 1) \qquad (2) \\ &= \lambda^3 + \frac{41}{3}\lambda^2 + \frac{304}{3}\lambda + 1440 \\ &= (\lambda + 13.85460)(\lambda - 0.09397 + 10.1945i)(\lambda - 0.09397 - 10.1945i) \qquad (3) \end{aligned}$$

(b)

(c) Using Eq(3), can you see what bifurcation is most likely to occur as ρ is reduced below 28 ? Using Eq(2), can you verify your conjecture and determine the value of $\rho = \rho_c$ at which the bifurcation occurs and what type of bifurcation it is. For $\rho < \rho_c$, choose initial states $\mathbf{x}_0$ (i) near $\mathbf{x}_e$ and (ii) far away from equilibrium and simulate both trajectories.

An interesting discussion of the *Lorenz* system can be found at
`http://en.wikipedia.org/wiki/Lorenz_system`

2. The *Jacobian* matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{bmatrix}$$

(a) Letting $u = \sqrt{c^2 - 4ab} \approx 5.6859$

(i)

$$\mathbf{x}_{e_+} = \begin{pmatrix} \frac{c+u}{2} \\ -\frac{c+u}{2a} \\ \frac{c+u}{2a} \end{pmatrix} = \begin{pmatrix} 5.6930 \\ -28.4649 \\ 28.4649 \end{pmatrix}, \qquad D\mathbf{f}(\mathbf{x}_{e_+}) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 28.4649 & 0 & -0.007026 \end{bmatrix}$$

$$\begin{aligned} \chi(\lambda) &= \lambda^3 - 0.1930\lambda^2 + 29.4635\lambda - 5.6859 \\ &= (\lambda - 0.1930)(\lambda + 0.000004596 - 5.4280i)(\lambda + 0.000004596 - 5.4280i) \end{aligned}$$

(ii)

$$\mathbf{x}_{e_-} = \begin{pmatrix} \frac{c-u}{2} \\ -\frac{c-u}{2a} \\ \frac{c-u}{2a} \end{pmatrix} = \begin{pmatrix} 0.007026 \\ -0.03513 \\ 0.03513 \end{pmatrix}, \qquad D\mathbf{f}(\mathbf{x}_{e_-}) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0.03513 & 0 & -5.6930 \end{bmatrix}$$

$$\begin{aligned} \chi(\lambda) &= \lambda^3 + 5.4930\lambda^2 - 0.1035\lambda + 5.6859 \\ &= (\lambda + 5.6870)(\lambda - 0.09700 + 0.9952i)(\lambda - 0.09700 - 0.9952i) \end{aligned}$$

(b)

(c) Concentrating on the fixed point near the origin ($\mathbf{x}_{e_-}$), the characteristic polynomial is

$$\chi(\lambda) = \lambda^3 - \left(a - \frac{c+u}{2}\right)\lambda^2 + \left(1 - a\frac{c+u}{2} + \frac{c-u}{2a}\right)\lambda + u.$$

Verify that a bifurcation occurs at $a = 0$. Can you identify it?

An interesting discussion of the *Rössler* system can be found at
`http://en.wikipedia.org/wiki/Rossler_attractor`