UNIVERSITY *of* LIMERICK

OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4315

SEMESTER: Autumn 2013

MODULE TITLE: Operations Research 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke/Dr. J. Kinsella

GRADING SCHEME:

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES:

- **Answer four questions correctly for full marks.**
- **Answer two questions from Q1–Q3 and two questions from Q4–Q6.**
- **50% of the marks are for the two questions from Q1–Q3 and 45% for the two questions from Q4–Q6.**

1 (a) (i) Explain briefly why the following strategy for the solution of I.P.'s is not useful: "Solve the L.P. relaxation then round off the components of the solution to the nearest integers". 1

(ii) Explain one method whereby a lower bound for the Standard Form I.P. $z = \max\{c^T x | Ax \leq b, x \geq 0, x \in \mathbb{Z}^n\}$ may be found. 1

(iii) Discuss the statement that "The Travelling Salesman Problem (TSP) is just the Assignment Problem (AP) with extra constraints." Be as specific as possible, defining both problems (objective function and constraints) algebraically. Explain clearly the role of the "subtour elimination" constraints. 3

(iv) When solving a TSP, of what use is an optimal solution to the AP obtained by dropping the subtour constraints? 1

(v) Explain carefully how L.P. Relaxation may be used to find an upper bound for z in a Standard Form I.P.. 1

(b) Consider the Integer Linear Program (IP):

$$\begin{aligned} \max 5x_1 + 2x_2 & \\ 5x_1 + x_2 & \leq 5/2 \\ 10x_1 + 6x_2 & \leq 45 \\ x_1, x_2 \leq 0 & \text{ and integer.} \end{aligned}$$

The corresponding Simplex Tableau (transforming the max problem into a min problem) is:

0	-5	-2	0	0
5/2	5	1	1	0
45	10	6	0	1

N.B. The Simplex Method and the Dual Simplex Method are stated on the last page of this paper.

(i) Apply one iteration of the Simplex Method and show that the Simplex Tableau now takes the form: 4

2.5	0	-1	1	0
0.5	1	0.2	0.2	0
40	0	4	-2	1

(ii) After a second iteration of the Simplex Method the Simplex Tableau now takes the form: (**N.B.do not perform the arithmetic!**)

5	5	0	2	0
2.5	5	1	1	0
30	-20	0	-6	1

Explain why this Tableau is optimal. 1

(iii) Explain why the solution to the LP Relaxation of the IP is $x_1 = 0, x_2 = 2.5$ and why we must now branch on x_2 and what are the two branches? 1

(iv) Consider the branch $S_0 : x_2 \leq 2$.

A. First show that the basic variable x_2 may be expressed in terms of the non-basic variables x_1 & x_3 as:
$x_2 = 2.5 - 5x_1 - x_3$. 1

B. Substitute this expression for x_2 into the S_0 branch constraint and show that it takes the form $-5x_1 - x_3 + s = -0.5$. (The variable s is the slack variable for the constraint $x_2 \leq 2$.) 1

C. Show that the Simplex Tableau with the addition of this constraint takes the form: 1

5	5	0	2	0	0
2.5	5	1	1	0	0
30	-20	0	-6	1	0
-0.5	-5	0	-1	0	1

(v) Explain why this tableau is optimal but infeasible. 1

(vi) Apply **one** iteration of the Dual Simplex Method (DSM) to this tableau and show that the Simplex Tableau now takes the form: 4

4.5	0	0	1	0	1
2	0	1	0	0	1
32	0	0	-2	1	-4
0.1	1	0	0.20	0	-0.2

(vii) Is this tableau LP optimal? Is it integer feasible? Explain. What is the solution to this LP relaxation? 1

(viii) What branching should you now make?

(ix) Having chosen a suitable branching, explain why the new tableau for the left-hand branch ($x_1 = 0$) is:

4.5	0	0	1	0	1
2	0	1	0	0	1
32	0	0	-2	1	-4
0.1	1	0	0.2	0	-0.2
-0.1	0	0	-0.2	0	0.2

A. What row and column is selected for pivoting with DSM? 1

B. Check that, after pivoting with DSM, x_1 and x_2 remain basic with $x_1 = 0$ and $x_2 = 2$. (No need to update the full tableau, just the top left-hand element and the second-last element in the the left-hand column.) and hence find the integer optimal value of z? 2

2 This Question is on the Branch & Bound Method for **Binary** Integer (B.I.P.'s) programs. Pseudo-code for the Look-ahead version of the Branch & Bound Method for BIP's is given at the end of this Paper.

(a) The B & B method for BIP's assumes that the problem is a minimisation (min) BIP with all non-negative cost coefficients, where the cost coefficients are non-decreasing from left to right. Demonstrate how this can be achieved using the following example: 4

$$\begin{aligned} \min 4x_1 + 2x_2 - 5x_3 + 7x_4 \\ -2x_1 - x_2 - 3x_3 - 3x_4 \geq -8 \\ 3x_1 - 2x_2 + x_3 + 2x_4 \geq 2 \\ x \in \mathbb{B}^4 \end{aligned}$$

(b) Given the transformed version of the BIP in part (a) (satisfying the above conditions, note the constant -5 in the objective)

$$\begin{aligned} \min 2x_1 + 4x_2 + 5x_3 + 7x_4 - 5 \\ -x_1 - 2x_2 + 3x_3 - 3x_4 \geq -5 \\ -2x_1 + 3x_2 - x_3 + 2x_4 \geq 1 \\ x \in \mathbb{B}^4 \end{aligned}$$

(i) Generate an upper bound z_U and a lower bound z_L on the optimal z using the look-ahead rule. 1

(ii) Is the all-zeroes point $x = (0, 0, 0, 0)$ feasible for the two constraints? Why? 1

(iii) If not, use the look-ahead rule to select a new x_L and z_L. Is it feasible for the two constraints? If it were, what would you conclude? Explain your reasoning. 2

(c) (i) Now branch on $x_1 = 0/1$. Draw the enumeration tree. 2

(ii) Find updated lower bounds for the sub-problems L_1 & R_1. 2

(iii) Check that neither node can be pruned by infeasibility. 2

(iv) Now check that one (you work out which) of $x_L^{(1)}$ or $x_R^{(1)}$ is feasible for the two constraints. 2

(v) Explain why this node can be pruned for optimality. 2

(vi) Update the value of the incumbent. 1

(vii) Update the enumeration tree. 2

(viii) Can you prune the remaining node by bound — explain carefully. 3

(ix) What is the optimal solution to the BIP? 1

3 In this Question you will solve the following DP.

- A shop must carry bags of salt for the approaching winter months.
- Initially there is no salt in stock.
- The shop buys salt from a wholesaler who delivers salt to the shop.
- **Bags of salt are delivered in boxes of five**.
- The shopkeeper has predicted the demand for each of the five winter months based on a (free) weather website.
- The Table gives
  - The maximum number of **bags** available for purchase at the start of each month
  - The predicted number of bags needed for each month.
- The shop can buy more than it needs at the start of a given month and use it later, incurring a storage cost of €1 per bag left over at the end of that month — payable at the beginning of the following month.
- The Table also shows the price of a bag of salt and the shipping cost per **bag** in each month .

Month	Demand (bags) d_n	Max. bags available	Shipping Cost s_n per bag	Wholesale price p_n per bag
1	10	20	€2	€2
2	20	25	€1	€3
3	15	20	€2	€3
4	20	25	€2	€4
5	10	15	€1	€3

Table 1: Data for salt orders

(a) (i) What are the stages/steps/iterations n of the process? 0.5

(ii) What are the states s of the system? 0.5

(iii) What are the decisions x_n to be made at each step n? 0.5

(iv) Define carefully $f_n(s)$ and $f_n(s, x_n)$. 0.5

(v) Write the recurrence relation for $f_n(s, x_n)$ in terms of $f_{n+1}(.)$ and the various costs. 1

(vi) For what value of s and n does does the shopkeeper wish to know the value of $f_n(s)$? 1

(b) Write the Table for the last of the 5 months ($n = 5$). 3

(c) Write the Table for the previous (fourth) month, carefully noting the demand, the maximum availibility for purchase and the fact that you may only buy bags in multiples of 5. 10

(d) Now, given the Tables for $n = 3$ & $n = 2$ below, compute the Table for $n = 1$ (with $s = 0$) and determine the minimum cost of meeting demand and **two** different strategies to achieve that minimum cost. 8

s	$x_2 = 0$	$x_2 = 5$	$x_2 = 10$	$x_2 = 15$	$x_2 = 20$	$x_2 = 20$	$f_2(s)$	x_2^*
0	*	*	*	*	315	315	315	15,20
5	*	*	*	300	300	300	300	15,20,25
10	*	*	285	285	285	285	285	10,15,20,25

Table 2: $n = 2$

s	$x_3 = 0$	$x_3 = 5$	$x_3 = 10$	$x_3 = 15$	$x_3 = 20$	$f_3(s)$	x_3^*
0	*	*	*	235	235	235	15,20
5	*	*	215	215	215	215	10,15,20
10	*	195	195	195	195	195	5,10,15,20
15	175	175	175	175	175	175	0,5,10,15,20

Table 3: $n = 3$

4 An American put option (right to sell a share) has a strike price of €100 with 3 periods left to maturity. The current share price is €102. Analysis of the market indicates that the share price volatility per period is described by: the stock will go up by €2 with probability 0.4, stay the same with probability 0.3, decrease by €2 with probability 0.3.

Use a Dynamic Programming formulation to price the option. Should this option be purchased for €0.3? Explain. 25

5 (a) In the context of a 2-person game, define what a *Nash* equilibrium is. 3

(b) What is a pure strategy ? What is a mixed strategy ? 3

(c) For strategies define strict dominance and weak dominance. 3

(d) By removing all strategies which are dominated by strict pure or mixed strategies, derive the reduced version of the following 2-player matrix game:

	D	E	F
A	(3,5)	(5,1)	(1,2)
B	(1,1)	(6,9)	(6,4)
C	(2,6)	(4,7)	(0,8)

8

(e) Derive the *Nash* equilibria and values of this game. 8

6 Consider the asymmetric duopoly game: Firm $i, \ i = 1, 2$ produces x_i items at a cost of

$$C(x_i) = \frac{1}{i}x_i + 20.$$

The items sell at a price of

$$p(x_1, x_2) = 5 - \frac{x_1 + x_2}{500}$$

each.

(a) Find the equilibrium of the game if it is played as a *Cournot* game, and prove that it is a *Nash* equilibrium. 8

(b) Find the equilibrium if it played as a *Stackelberg* game with Firm 1 as leader. 8

(c) Contrast and comment on the two solutions. 4

(d) Firm 1 receives an injection of capital and initiates a "leader strategy". However Firm 2 persists with its *Cournot* strategy. What happens and what should each firm do in future interactions? 5

Algorithm 1 (Simplex Method)

begin (Start with a Canonical tableau s.t. $\mathbf{b} \geq \mathbf{0}$.)
while NOT finished do
if $c_j \geq 0$ for all j
then **STOP** (Tableau is optimal.)
else Select j s.t. $c_j < 0$.
fi
if $a_{ij} \leq 0$ for all $i = 1, \ldots, m$
then **STOP** (Problem is unbounded.)
fi
Select k such that:
$\frac{b_k}{a_{kj}} = \min_i \left\{ \frac{b_i}{a_{ij}} \text{such that } a_{ij} > 0 \right\}$ (k attains the min.)
Pivot on a_{kj}. (Divide Row k across by a_{kj} and add
end ... multiples of Row k to the rows above & below
end ... ... introducing zeros into column j.)

Algorithm 2 (Dual Simplex Method)

begin (Start with a tableau s.t. $\mathbf{c} \geq \mathbf{0}$.)
while NOT finished do
if $b_i \geq 0$ for all i
then **STOP** (Tableau is optimal.)
else Select i s.t. $b_i < 0$.
fi
if $-a_{ij} \leq 0$ for all $j = 1, \ldots, n$
then **STOP** (Dual unbounded $\equiv$ Primal infeasible.)
fi
Select k such that:
$\frac{c_k}{a_{ik}} = \max_j \left\{ \frac{c_j}{a_{ij}} \text{such that } a_{ij} < 0 \right\}$ (k attains max.)
Pivot on a_{ik}. (Divide Row k across by a_{ik} and add
end ... multiples of Row k to the rows above & below
end ... ... introducing zeros into column i.)

Algorithm 3 (Look-ahead Binary Branch & Bound)

(1) begin
(2) **Initialise** : if $x = 0$ is feasible then $x = 0$ is opt, STOP fi
(3) Set bounds on z^*: $z_U = \sum_j c_j$, $z_L = z_1$, $x_L = (1, 0, \ldots, 0)$
(4) Initialise NodeList to x_L.
(5) if x_L is feasible then x_L is opt, STOP fi
(6) $k \leftarrow 1. NodeListEmpty := FALSE$
(7) while $\neg NodeListEmpty$ do
(8) begin
(9) **Branch**: Select a remaining subset of feasible solutions (node)
(10) (the whole feasible set at the first iteration) and partition the subset
(11) into two smaller subsets by adding constraints $x_k = 0$ and $x_k = 1$.
(12) **Bound**: For each new subset, set x_L to the completion having its
(13) $(k+1)$st component $= 1$ and all subsequent components $= 0$.
(14) Use x_L to determine a lower bound z_L on z over that subset.
(15) **Prune/fathom**: Examine each unpruned node and prune it if
(16) $z_L \geq z_U$ **OR** one or more constraints cannot be satisfied
(17) by any completion in the subset **OR** x_L is feasible.
(18) if x_L feasible then x_L is new **incumbent**, $z_U \leftarrow z_L$.
(19) GOTO (15) and Check if other nodes can be pruned.
(20) fi
(21) **Test**:if no unpruned nodes left then $STOP$, **incumbent** is opt. else $k \rightarrow k + 1$. fi
(22) end
(23) end