UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4315

SEMESTER: Autumn 2015

MODULE TITLE: Operations Research 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURERS: Burke/Kinsella/O'Brien

GRADING SCHEME: 95%

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES:

- **Answer four questions correctly for full marks.**
- **Answer two questions from Q1–Q3 and two questions from Q4–Q6.**
- **45% of the marks are for the two questions from Q1–Q3 and 50% for the two questions from Q4–Q6.**

1 (a) In terms of a strategic (matrix) game, what is a dominated strategy? Describe the technique of iterated elimination of dominated strategies. What is the technique used for ? 5

(b) By removing all strategies which are dominated by strict pure or mixed strategies, derive the reduced version of the following 2-player zero-sum matrix game:

	D	E	F
A	(5,-5)	(1,-1)	(2,-2)
B	(1,-1)	(0,0)	(3,-3)
C	(2,-2)	(3,-3)	(6,-6)

8

(c) In terms of a 2-player zero-sum game, what is a minimax strategy? 3

(d) Derive the minimax strategies and value of the above game. 9

2 In a game show, contestants S´ile and Seán start the last round with €500 and €400 respectively. Each must decide to pass or play. If a player passes, they keep their money but if playing the first person gains €200 with probability 1/2 or loses all their money with probability 1/2, while the second person gains or loses €200, each with probability 1/2. These outcomes are independent of each other. The player with the most money at the end of the round gets a bonus of €200.

(a) If S´ile goes first and Seán sees her move, draw the game tree. 9

(b) Show that the strategic form of the game is

	Pass	Play
Pass	(7,4)	(6,5)
Play	$\left(\frac{9}{2},5\right)$	$\left(\frac{9}{2},5\right)$

where payoffs are expected values in 00's. 8

(c) Solve the game. 8

3 Consider the following asymmetric duopoly game with isoelastic demand: Two firms sell equivalent items at a price of

$$p(x_1, x_2) = \frac{600}{x_1 + x_2}$$

per item, where Firm i, $(i = 1, 2)$, produces x_i items at a cost of

$$C(x_i) = m_i x_i.$$

The marginal costs are given by $m_i = i + 1$ respectively.

(a) If the game is played as a *Cournot* game, show that the best response $x_i = B(x_j)$ of Firm i to Firm j is given by

$$B(x_j) = \sqrt{\frac{600x_j}{m_i}} - x_j$$

and hence find the *Nash* equilibrium. 8

(b) Find the equilibrium if it played as a *Stackelberg* game with Firm 1 as leader. 8

(c) Contrast and comment on the two solutions. 4

(d) Firm 1 decides it will stick with the "leader strategy" in further production cycles. However Firm 2 decides to ignore this and persist with its *Cournot* strategy of part(a). What happens and what should each firm do in future interactions? 5

4 (a) (i) Explain briefly why the following strategy for the solution of Integer Linear Programs (IP's) is not useful: "Substitute all integer values in the range $[0, 100]$ (say) for each independent variable x_i. For each feasible vector **x**, record the corresponding z–value. Then pick the feasible **x** which maximises z." 1%

(ii) Explain briefly why the following strategy for the solution of Integer Linear Programs (IP's) is not useful: "Solve the LP relaxation then round off the components of the solution to the nearest integers". 1%

(iii) Explain carefully how lower bounds for the Standard Form IP $z = \max\{c^T x | Ax \leq b, x \geq 0, x \in \mathbb{Z}^n\}$ may be found. 1%

(iv) Explain carefully how LP Relaxation may be used to find an upper bound for z in a Standard Form IP. 2%

(b) Given an LP (the **Primal** problem) we can write a closely related LP, its **Dual**:

$$\begin{aligned} z &= \max\{c^T x : Ax \leq b, x \in \setminus^n, x \geq 0\} \quad \textbf{Primal} \\ w &= \min\{b^T y : A^T y \geq c, y \in \setminus^m, y \geq 0\}. \quad \textbf{Dual} \end{aligned}$$

(i) Prove the Weak Duality Theorem: for **any** primal feasible point x and **any** dual feasible point y, $b^T y \geq c^T x$. 4%

(ii) Show that the Weak Duality Theorem implies that **any** feasible point y for the Dual problem gives an upper bound for the optimal solution z of the original IP, namely $z \leq b^T y$. 4%

(c) OBKB Investments is considering investments into 5 projects: A, B, C, D, and E.

Each project has an initial cost, an expected profit rate (one year from now) expressed as a percentage of the initial cost, and an associated risk of failure. These numbers are given in the table below:

	A	B	C	D	E
Initial Cost	1.5	0.9	0.7	1.5	2.1
Profit Rate	10%	15%	10%	12%	10%
Failure Risk	6%	4%	6%	5%	4%

(i) Provide a formulation to choose the projects that maximize total expected profit, such that OBKB Investments does not invest more than 5M dollars and its average failure risk is not over 5%.
For example, if OBKB Investments invests only into A and B, it invests only 2.5M dollars and its average failure risk is $(6\% + 4\%)/2 = 5\%$. 3%

(ii) Suppose that if A is chosen, B must be chosen. Modify your formulation. 3%

(iii) Suppose that if C is chosen, E must not be chosen. Modify your formulation. 3%

(iv) Suppose that if A and C are chosen, D must be chosen. Modify your formulation. 3%

5 Consider the IP:

$$\begin{aligned} \max x_1 + 2x_2 \\ -x_1 + x_2 \leq 10 \\ 15x_1 + 16x_2 \leq 240 \\ x_1, x_2 \geq 0 \text{ and integer.} \end{aligned}$$

You will be asked to partly solve the IP, using the Branch and Bound Method. **You should draw an enumeration tree/diagram to keep track of your progress — draw the enumeration tree on an otherwise blank page.**

The corresponding Simplex Tableau (transforming the max problem into a min problem) is:

$$T_0 = \begin{array}{|r|rrrr|} \hline 0 & -1 & -2 & 0 & 0 \\ \hline 10 & -1 & 1 & 1 & 0 \\ 240 & 15 & 16 & 0 & 1 \\ \hline \end{array}$$

N.B. The Simplex Method and the Dual Simplex Method are stated on the last page of this paper.

You will partly solve the IP, following the steps on the next pages — remember to draw and fill in an enumeration tree.

(a) Apply one iteration of the Simplex Method to the starting tableau (T_0) and show that the Simplex Tableau now takes the form: 3%

20	−3	0	2	0
10	−1	1	1	0
80	31	0	−16	1

(b) After a second iteration of the Simplex Method the Simplex Tableau now takes the form: (**N.B. Do not perform the arithmetic!**)

$T_2 =$

27.74	0	0	0.45	0.10
12.58	0	1	0.48	0.03
2.58	1	0	−0.52	0.03

(i) Explain why this Tableau is LP optimal. 1%

(ii) What is the solution to the LP Relaxation of the IP? 1%

(iii) Why is this not IP-optimal? 1%

(iv) What bound can you conclude for the optimal value of the objective for the original IP? 1%

(v) What are the two possible branches on x_1 and what are the two possible branches on x_2? 1%

(c) Consider the **right** branch corresponding to a lower bound on x_2 ($x_2 \geq$?).

(i) The LP-optimal tableau T_2 must be amended by adding an extra row & column corresponding to the extra constraint. Explain carefully why the new tableau T_3 (with 4 rows and 6 columns) takes the form: 2%

$T_3 =$

27.74	0	0	0.45	0.10	0
12.58	0	1	0.48	0.03	0
2.58	1	0	−0.52	0.03	0
−13	0	−1	0	0	1

(ii) Which element of T_3 should you pivot on next? Explain carefully. 1%

(iii) The resulting tableau is

$T_4 =$

27.74	0	0	0.45	0.10	0
12.58	0	1	0.48	0.03	0
2.58	1	0	−0.52	0.03	0
−0.42	0	0	0.48	0.03	1

(iv) Explain carefully why T_4 is infeasible and update your enumeration tree. 1%

(d) Consider the **left** branch corresponding to an upper bound on x_2 ($x_2 \leq$?).

(i) The LP-optimal tableau T_2 must be amended by adding an extra row & column corresponding to the extra constraint. Explain carefully why the new tableau T_5 (with 4 rows and 6 columns) takes the form: 2%

$T_5 =$

27.74	0	0	0.45	0.10	0
12.58	0	1	0.48	0.03	0
2.58	1	0	−0.52	0.03	0
12	0	1	0	0	1

(ii) Which element of T_5 should you pivot on next? Explain carefully. 1%

(iii) The resulting tableau is

$T_6 =$

27.74	0	0	0.45	0.10	0
12.58	0	1	0.48	0.03	0
2.58	1	0	−0.52	0.03	0
−0.58	0	0	−0.48	−0.03	1

Using the Dual Simplex Method, which element should you pivot on next so that the tableau is LP-optimal? Explain carefully. 2%

(iv) The resulting tableau is:

$T_7 =$

27.20	0	0	0	0.07	0.93
12	0	1	0	0	1
3.20	1	0	0	0.07	−1.07
1.20	0	0	1	0.07	−2.07

(v) Is T_7 LP-optimal? Explain carefully. 1%

(vi) Is T_7 IP-optimal? Explain carefully. 1%

(e) Consider the **left** branch corresponding to an upper bound on x_1 ($x_1 \leq$?).

(i) As T_7 is not IP-optimal it must be amended by adding an extra row & column corresponding to the extra constraint. Explain carefully why the new tableau T_8 (with 5 rows and 7 columns) takes the form: 2%

$T_8 =$

27.20	0	0	0	0.07	0.93	0
12	0	1	0	0	1	0
3.20	1	0	0	0.07	−1.07	0
1.20	0	0	1	0.07	−2.07	0
3	1	0	0	0	0	1

(ii) Which element of T_8 should you pivot on next? Explain carefully. 1%

(iii) The resulting tableau is

$T_9 =$

27.20	0	0	0	0.07	0.93	0
12	0	1	0	0	1	0
3.20	1	0	0	0.07	−1.07	0
1.20	0	0	1	0.07	−2.07	0
−0.20	0	0	0	−0.07	1.07	1

(iv) Using the Dual Simplex Method, which element of T_9 should you pivot on next? Explain carefully. 1%

(v) The resulting tableau is:

27	0	0	0	0	2	1
12	0	1	0	0	1	0
3	1	0	0	0	0	1
1	0	0	1	0	−1	1
3	0	0	0	1	−16	−15

(vi) Does this tableau represent the optimal solution to the IP? Explain carefully why or why not. 1%

(vii) What are the solution $\mathbf{x}$ and the objective value z^* associated with the tableau? 1%

6 An IP is to be solved using the tabular Branch and Bound method.

Use the solution grids below to solve the problem. Each node is referenced by its tree level, ordered from left to right so that the annotation **Node XY** is the node at level **X** at position **Y** where **Y** $= 1$ is the left-most position in level **X**. 25%

	Node 0		Node 1A		Node 1B
(i)	$X = 6.50, Y = 7.50$	(i)	$X = 6.00, Y = 5.00$	(i)	$X = 5.75, Y = 7.00$
(ii)	$X = 7.50, Y = 6.00$	(ii)	$X = 6.00, Y = 6.00$	(ii)	$X = 6.625, Y = 6.00$
(iii)	$X = 7.20, Y = 5.40$	(iii)	$X = 4.50, Y = 5.50$	(iii)	$X = 5.625, Y = 6.875$
(iv)	$X = 6.00, Y = 9.00$	(iv)	$X = 7.00, Y = 5.00$	(iv)	$X = 7.50, Y = 4.125$
(v)	$X = 7.00, Y = 5.75$	(v)	$X = 7.00, Y = 3.00$	(v)	$X = 7.00, Y = 5.75$
	Node 2A		Node 2B		Node 2C
(i)	$X = 6.50, Y = 4.50$	(i)	$X = 7.25, Y = 5.75$	(i)	$X = 6.00, Y = 5.00$
(ii)	$X = 6.25, Y = 5.50$	(ii)	$X = 6.25, Y = 5.50$	(ii)	$X = 6.625, Y = 5.75$
(iii)	$X = 5.625, Y = 5.00$	(iii)	$X = 7.25, Y = 5.25$	(iii)	$X = 5.625, Y = 6.75$
(iv)	$X = 7.50, Y = 4.75$	(iv)	$X = 7.00, Y = 4.00$	(iv)	$X = 6.00, Y = 6.00$
(v)	$X = 7.25, Y = 5.25$	(v)	$X = 7.50, Y = 4.50$	(v)	$X = 6.00, Y = 6.416$
	Node 2D		Node 3A		Node 3B
(i)	$X = 5.75, Y = 7.00$	(i)	$X = 6.125, Y = 2.50$	(i)	$X = 6.75, Y = 3.50$
(ii)	$X = 7.00, Y = 7.00$	(ii)	$X = 6.75, Y = 3.50$	(ii)	$X = 6.25, Y = 5.50$
(iii)	$X = 5.625, Y = 6.75$	(iii)	$X = 6.8, Y = 3.666$	(iii)	$X = 6.125, Y = 2.50$
(iv)	$X = 7.50, Y = 4.00$	(iv)	$X = 7.2, Y = 2.9$	(iv)	$X = 6.00, Y = 4.50$
(v)	$X = 7.00, Y = 6.00$	(v)	$X = 7.00, Y = 3.875$	(v)	$X = 7.00, Y = 3.875$

(**See next page for remainder of table.**)

	Node 3C		Node 3D		Node 3E
(i)	$X = 5.75, Y = 6.75$	(i)	$X = 7.25, Y = 3.333$	(i)	$X = 6.00, Y = 6.00$
(ii)	$X = 5.625, Y = 7.333$	(ii)	$X = 4.833, Y = 6.5$	(ii)	$X = 6.125, Y = 5.75$
(iii)	$X = 7.50, Y = 7.125$	(iii)	$X = 3.75, Y = 8.666$	(iii)	$X = 6.00, Y = 6.75$
(iv)	$X = 7.125, Y = 7.00$	(iv)	$X = 5.25, Y = 4.33$	(iv)	$X = 5.75, Y = 5.00$
(v)	$X = 7.00, Y = 8.00$	(v)	$X = 5, Y = 5.50$	(v)	$X = 7.00, Y = 6.00$
	Node 3F		Node 3G		Node 3H
(i)	$X = 5.75, Y = 7.00$	(i)	$X = 5.00, Y = 6.00$	(i)	$X = 6.75, Y = 3.50$
(ii)	$X = 7.00, Y = 7.00$	(ii)	$X = 6.75, Y = 3.50$	(ii)	$X = 6.75, Y = 2.10$
(iii)	$X = 5.125, Y = 7.00$	(iii)	$X = 6.75, Y = 4.50$	(iii)	$X = 6.75, Y = 3.50$
(iv)	$X = 6.00, Y = 6.00$	(iv)	$X = 6.75, Y = 4.50$	(iv)	$X = 6.00, Y = 4.00$
(v)	$X = 7.25, Y = 5.00$	(v)	$X = 3.666, Y = 8.50$	(v)	$X = 7.00, Y = 3.50$

Algorithm 1 (Simplex Method)

begin (Start with a Canonical tableau s.t. $\mathbf{b} \geq \mathbf{0}$.)
while NOT finished do
if $c_j \geq 0$ for all j
then **STOP** (Tableau is optimal.)
else Select j s.t. $c_j < 0$.
fi
if $a_{ij} \leq 0$ for all $i = 1, \ldots, m$
then **STOP** (Problem is unbounded.)
fi
Select k such that:
$\frac{b_k}{a_{kj}} = \min_i \left\{ \frac{b_i}{a_{ij}} \text{such that } a_{ij} > 0 \right\}$ (k attains the min.)
Pivot on a_{kj}. (Divide Row k across by a_{kj} and add
end ... multiples of Row k to the rows above & below
end ... ... introducing zeros into column j.)

Algorithm 2 (Dual Simplex Method)

begin (Start with a tableau s.t. $\mathbf{c} \geq \mathbf{0}$.)
while NOT finished do
if $b_i \geq 0$ for all i
then **STOP** (Tableau is optimal.)
else Select i s.t. $b_i < 0$.
fi
if $-a_{ij} \leq 0$ for all $j = 1, \ldots, n$
then **STOP** (Dual unbounded $\equiv$ Primal infeasible.)
fi
Select k such that:
$\frac{c_k}{a_{ik}} = \max_j \left\{ \frac{c_j}{a_{ij}} \text{such that } a_{ij} < 0 \right\}$ (k attains max.)
Pivot on a_{ik}. (Divide Row k across by a_{ik} and add
end ... multiples of Row k to the rows above & below
end ... ... introducing zeros into column i.)