UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4328

SEMESTER: Spring 2011

MODULE TITLE: Mathematical Control Theory

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 100 %

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 4 questions. There are notes & formulae supplied.**

Unless otherwise stated, the following notation is used throughout: $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{y}$ are the $n \times 1$ state, $r \times 1$ input and $m \times 1$ output vectors respectively, whose components are real functions of time. $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}_i(\mathbf{x}),\ i = 1, 2, \ldots r$ are $n \times 1$ C^1 vector fields. A is a $n \times n$ state matrix, B a $n \times r$ input matrix and C a $m \times n$ output matrix, all of whose elements are real numbers. LTI is an acronym for linear time invariant.

1. (a) Define complete controllability and complete observability for the LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}$$

4

(b) By using the modal transformation determine whether the system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix}.$$

is completely controllable and/or completely observable. 8

(c) State the *Kalman* canonical structure theorem. Classify the modes of the system of part (b) according to the theorem. 4

(d) Define stabilisability and detectability. Is the system of part (b) stabilisable, detectable ? 4

2. (a) Define stability and asymptotic stability in the sense of *Lyapunov* for the continuous-time dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}).$$

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(b) Describe *Lyapunov*'s 2nd or Direct Method, clearly stating the relevant theorems. 3

(c) State *La Salle*'s Invariance Principle and discuss how it may be used to determine asymptotic stability. 3

(d) Determine whether the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -2x_2|x_2| - x_1 \cos x_1 - x_1^5$$

is an asymptotically stable fixed point. Is it globally asymptotically stable? 6

(e) Can the control u be chosen to stabilise the system

$$\ddot{x} + \beta\dot{x} + \alpha x^2 = u$$

where $-3 < \beta < -1$ and $|\alpha| < 4$? If so, suggest a possible control; if not, prove it. 6

3. (a) State the pole placement problem for the LTI discrete-time system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$

with linear state feedback

$$\mathbf{u}_k = K\mathbf{x}_k + \mathbf{v}_k.$$

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(b) Derive an algorithm for placing the poles of a completely controllable single input system using linear state feedback at specified locations in the complex plane. 6

(c) Hence find a linear state feedback controller that places the unstable pole(s) of the open loop system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

at $\lambda = 1/2$ in the complex plane. 8

(d) If the complete state variable is not available for measurement but only the output $y_k = x_1(k) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \mathbf{x}_k$, is it possible to design a *Luenberger* observer to estimate the state variables? If so, can the output of this state estimator be used in place of the actual state to implement the feedback control designed in part (c), and in what sense are the two control strategies equivalent? 4

4. (a) A square system is one in which the number of inputs equals the number of outputs. For the square LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}$$

define the decoupling indices and derive the conditions under which the system may be "integrator-decoupled" by linear state feedback

$$\mathbf{u} = K\mathbf{x} + M\mathbf{v}$$

6

(b) Can the system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

be integrator-decoupled ? 6

(c) What is the remnant of an integrator-decoupled system, and why is it important ? Is integrator-decoupling an appropriate design tool for the system of part (b) ? 8

5. (a) For the affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{r} \mathbf{g}_i(\mathbf{x})u_i, \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}),$$

define what is meant by a *control-Lyapunov function*. 4

(b) Prove that $V(\mathbf{x}) = x_1^2 + 3x_2^2$ is not a control-*Lyapunov* function for the system

$$\dot{\mathbf{x}} = \begin{pmatrix} x_2^2 \\ x_1 + x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

By choosing an appropriate value for β, show that the function $V(\mathbf{x}) = x_1^2 + \beta x_1 x_2 + 3x_2^2$ can constitute a control-*Lyapunov* function for the given system. 10

(c) Derive (i) a control based on cancellation of nonlinearities and (ii) one based on *Sontag*'s formula which will stabilise the system of part (b). Which control is preferable and why? 6

6. (a) By solving the continuous-time Algebraic *Riccati* Equation or otherwise, find the stabilising feedback control that minimises the cost functional

$$J(u) = \int_0^\infty 3x_1^2 + 2x_2^2 + u^2 \, dt$$

for the LTI system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

10

(b) Using the *Hamilton-Jacobi-Bellman* equation, show that the stabilising feedback that minimises the cost functional

$$J(u) = \int_0^\infty 4x^2 + u^2 \, dt$$

for the system

$$\dot{x} = \sqrt{|x|} + u$$

is given by

$$u = u^* \triangleq -\sqrt{|x|} - x\sqrt{4 + \frac{1}{|x|}}.$$

10

7. (a) State the conditions under which the single input affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

can be transformed by *feedback linearisation* to

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \mathbf{z} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v$$

and give an algorithm for doing so. 4

(b) Find the feedback linearisation scheme which transforms the system

$$\dot{\mathbf{x}} = \begin{pmatrix} x_2 \\ x_1^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

to the linear form of part (a). 10

(c) Find the control u that will result in the linear transformed system of part (b) having the characteristic equation $\lambda^2 + 4\lambda + 3 = 0$. 6