UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4328

SEMESTER: Spring 2016

MODULE TITLE: Mathematical Control Theory

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions. There are notes & formulae supplied.**

Unless otherwise stated, the following notation is used throughout: $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{y}$ are the $n \times 1$ state, $r \times 1$ input and $m \times 1$ output vectors respectively, whose components are real functions of time. $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}_i(\mathbf{x}),\ i = 1, 2, \ldots r$ are $n \times 1$ C^1 vector fields. A is a $n \times n$ state matrix, B a $n \times r$ input matrix and C a $m \times n$ output matrix, all of whose elements are real numbers. LTI is an acronym for linear time invariant.

1. (a) Define complete controllability and complete observability for the LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}.$$

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(b) By using the *Popov-Belevitch-Hautus* (PBH) tests determine whether the system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$$

is completely controllable and/or completely observable. 7

(c) State the *Kalman* canonical structure theorem. Classify the modes of the system of part (b) according to the theorem. 3

(d) Define stabilisability and detectability. Is the system of part (b) stabilisable, detectable ? 3

2. (a) State *La Salle*'s Invariance Principle and discuss how it may be used to establish asymptotic stability of the origin for the continuous-time system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}).$$

3

(b) Using the Invariance Principle or otherwise, determine whether the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sinh x_2 - c(x_1)$$

where

$$c(x_1) = \begin{cases} x_1 e^{x_1}, & \text{if } x_1 \leq 0, \\ x_1 e^{-x_1}, & \text{if } 0 < x_1 \end{cases}$$

is an asymptotically stable fixed point. Is it globally asymptotically stable? 5

(c) Can the control u be chosen to globally stabilise the origin of the uncertain system

$$\ddot{x} + \beta\dot{x} + \alpha x^2 = u$$

where $1 < \beta < 3$ and $-2 < \alpha < 2$? If so, suggest a possible control; if not, prove it. 8

3. (a) State the pole placement problem for the LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

with linear state feedback

$$\mathbf{u} = K\mathbf{x} + \mathbf{v}$$

and give a necessary and sufficient condition under which it can be solved. 4

(b) Find a linear state feedback controller that places the unstable pole(s) of the open loop system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

at $\lambda = -1$ in the complex plane. 8

(c) If the complete state variable is not available for measurement but only the output $y = x_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{x}$, prove that is it possible to design a *Luenberger* observer to estimate the state variables. What is the minimum possible order of such an observer ? 4

4. (a) A square system is one in which the number of inputs equals the number of outputs. For the square LTI system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}$$

define the decoupling indices and relative degrees. State the conditions under which the system may be "integrator-decoupled" by linear state feedback

$$\mathbf{u} = H\mathbf{x} + M\mathbf{v}$$

3

(b) Can the system with

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

be integrator-decoupled ? 5

(c) What is the remnant of an integrator-decoupled system, and why is it important ? Is integrator-decoupling an appropriate design tool for the system of part (b) ? 8

5. (a) For the affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{r} \mathbf{g}_i(\mathbf{x}) u_i, \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}),$$

define what is meant by a *control-Lyapunov function*. 3

(b) For what range of values of α is

$$V(\mathbf{x}) = x_1^2 + 2\alpha x_1 x_2 + x_2^2$$

a control-*Lyapunov* function for the system

$$\dot{\mathbf{x}} = \begin{pmatrix} x_2 \\ x_1 \cosh x_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

10

(c) State *Artstein*'s Theorem, and use the equivalent *Sontag*'s formula to find a stabilising control for the system and *control-Lyapunov function* of part (b) using an admissible value for α. 3

6. For the nonlinear continuous-time system

$$\dot{x} = x^2 + u$$

(a) by using the *Hamilton-Jacobi-Bellman* equation, show that the stabilising feedback control that minimises the cost functional

$$J(u) = \int_0^\infty x^2 + xu + u^2 \, dt$$

for the system is given by

$$u = u^* \triangleq -x^2 - x\sqrt{(x - 1/2)^2 + 3/4}.$$

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(b) For the system, design a control by feedback linearisation (cancellation of nonlinearities) which transforms the system to the linear system

$$\dot{x} = -kx$$

and then places the closed-loop pole of this linear system at $\lambda = -1$. 2

(c) Comment on the designs and how they are related. 4

7. A controlled crop - insect (pest) system with normalised crop biomass and pest population size of c and p respectively is described by

$$\begin{aligned}\dot{c} &= 2c(1-c) - 2cp \\ \dot{p} &= cp - (1/2)p + u\end{aligned}$$

where u is the rate of insecticide application.

(a) The system has an unstable equilibrium at $c = 1, p = 0$ corresponding to no application of insecticide. By using the change of variables $x_1 = c - 1,\ x_2 = p$ show that the system equations can be written as

$$\begin{aligned}\dot{x}_1 &= -2(1 + x_1)(x_1 + x_2) \\ \dot{x}_2 &= (1/2 + x_1)x_2 + u\end{aligned}$$

2

(b) Show that the system of part (b) has a relative degree of 2 if the output $y = x_1$ and hence find the corresponding state transformation and linearising control that transforms the system to *normal form*. 8

(c) Find the feedback control that places the poles of the exact linearised system at $\lambda = -2$. 5