1.

$$U = [B, AB, A^2B] = \begin{pmatrix} 1 & 0 & , & 0 & 0 & , & 0 & 0 \\ 2 & 2 & , & -1 & -4 & , & 2 & 8 \\ 1 & 2 & , & -1 & -4 & , & 2 & 8 \end{pmatrix}$$

which has rank $= 2$. So system is not CC. A has eigenvalues $0, -1, -2$ with corresponding eigenvectors $(2,3,1)^T$, $(0,2,1)^T$ and $(0,1,1)^T$ respectively. With the modal transformation $\mathbf{x} = E\mathbf{z}$ where

$$E = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

the transformed system becomes $\dot{\mathbf{z}} = \Lambda\mathbf{x} + \tilde{B}\mathbf{u}$, $\mathbf{y} = \tilde{C}\mathbf{z}$ where

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\tilde{B} = E^{-1}B = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & -1 \\ \frac{1}{2} & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ \frac{1}{2} & 2 \end{pmatrix}$$

and

$$\tilde{C} = CE = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

Since the middle row of $\tilde{B}$ is zero, we see that the corresponding mode $\lambda = -1$ is uncontrollable. Since $\lambda = -1$ is asymptotically stable, the system is stabilisable. Since the first column of $\tilde{c}$ is zero, we see that the corresponding mode $\lambda = 0$ is unobservable. Since $\lambda = 0$ is **not** asymptotically stable, the system is not detectable.

2. (a)

$$U = [b, Ab] = \begin{pmatrix} 3 & 11 \\ 2 & 7 \end{pmatrix}$$

which has rank $= 2$. So system is CC. A has characteristic polynomial $\lambda^2 - 3\lambda + 2$. The transformed equations are therefore

$$\dot{\hat{\mathbf{x}}} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \hat{\mathbf{x}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

with corresponding controllability matrix

$$\hat{U} = [\hat{b}, \hat{A}\hat{b}] = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

Hence

$$\hat{U}^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}$$

and the required transformation is $\mathbf{x} = \hat{T}\hat{\mathbf{x}}$ where

$$\hat{T} = U\hat{U}^{-1} = \begin{pmatrix} 3 & 11 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

(b)

$$U = [b, Ab] = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

which has rank = 1. So system is not CC.

(c)

$$U = [b, Ab, A^2b] = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -11 \end{pmatrix}$$

which has rank = 3. So system is CC. A (in companion form) has characteristic polynomial $\lambda^3 + 6\lambda^2 + 11\lambda + 6$. The transformed equations are therefore

$$\dot{\hat{\mathbf{x}}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} \hat{\mathbf{x}} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

with corresponding controllability matrix

$$\hat{U} = [\hat{b}, \hat{A}\hat{b}, \hat{A}^2\hat{b}] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{pmatrix}$$

Hence

$$\hat{U}^{-1} = \begin{pmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the required transformation is $\mathbf{x} = \hat{T}\hat{\mathbf{x}}$ where

$$\hat{T} = U\hat{U}^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -11 \end{pmatrix} \begin{pmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -10 & -6 & -1 \\ 6 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix}$$

3. We define S_j as the set of states that are null Ω - controllable in at most j time steps, and ΔS_j as the set of states that can be controlled to S_j in 1 time step. Hence

$$S_{j+1} = S_j \cup \Delta S_j$$

(a)

$$x_{k+1} = 2x_k + u_k, \qquad\qquad -4 \leq u_k \leq 4$$

S_1 is obtained by solving

$$0 = 2x + u \Rightarrow x = -\frac{u}{2} \in [-2,\ 2] \triangleq S_1$$

Similarly ΔS_1 is obtained by solving

$$S_1 \ni x' = 2x + u \Rightarrow x = \frac{x' - u}{2} \in [-1 - 2,\ 1 + 2] = [-3,\ 3] = \Delta S_1$$

Note that this last set contains S_1 and thus we get $S_2 = [-3,\ 3]$. More generally, assuming that $S_j = [-a_j,\ a_j]$; then

$$S_j \ni x' = 2x + u \Rightarrow x = \frac{x' - u}{2} \in \left[-\frac{a_j}{2} - 2,\ \frac{a_j}{2} + 2\right] = \Delta S_j$$

Again this last set contains S_j and so we have

$$S_{j+1} \triangleq [-a_{j+1},\ a_{j+1}] = \left[-\frac{a_j}{2} - 2,\ \frac{a_j}{2} + 2\right]$$

. Thus we have the recursion

$$a_{j+1} = \frac{1}{2}a_j + 2$$

In the limit as $j \to \infty$, this gives $a_\infty = 4$. Hence we get

$$S_\infty = (-4,\ 4)$$

Why is there an open interval rather than a closed interval in the answer?

(b)

$$x_{k+1} = 2x_k + u_k, \qquad -2 \leq u_k \leq -1$$

S_1 is again obtained by solving

$$0 = 2x + u \Rightarrow x = -\frac{u}{2} \in \left[\frac{1}{2},\ 1\right] = S_1$$

Similarly ΔS_1 is obtained by first solving

$$S_1 \ni x' = 2x + u \Rightarrow x = \frac{x' - u}{2} \in \left[\frac{1}{4} + \frac{1}{2},\ \frac{1}{2} + 1\right] = \left[\frac{3}{4},\ \frac{3}{2}\right] = \Delta S_1$$

Hence $S_2 = S_1 \cup \Delta S_1 = \left[\frac{1}{2},\ \frac{3}{2}\right]$.
More generally, assuming that $S_j = [\frac{1}{2},\ a_j]$; then

$$S_j \ni x' = 2x + u \Rightarrow x = \frac{x' - u}{2} \in \left[\frac{1}{4} + \frac{1}{2},\ \frac{a_j}{2} + 1\right] = \Delta S_j$$

Again $S_{j+1} = S_j \cup \Delta S_j = \left[\frac{1}{2},\ \frac{a_j}{2} + 1\right] \triangleq \left[\frac{1}{2},\ a_{j+1}\right]$. Thus we have the recursion

$$a_{j+1} = \frac{1}{2}a_j + 1$$

In the limit as $j \to \infty$, this gives $a_\infty = 2$. Hence we get

$$S_\infty = \left[\frac{1}{2},\ 2\right)$$

(c)

$$\mathbf{x}_{k+1} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{5}{2} \end{pmatrix} \mathbf{x}_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k, \qquad | u_k | \leq 1$$

One method, if applicable, of dealing with a higher than first order system is to transform it to modal form and find the set of null controllable values for each mode. For this example, the transformed system [$E = \begin{pmatrix} 1 & 1 \\ 1/2 & 2 \end{pmatrix}$] is

$$\mathbf{z}_{k+1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \mathbf{z}_k + \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} u_k, \qquad -1 \leq u_k \leq 1$$

or using slightly different notation

$$\begin{aligned} z_1(k+1) &= \frac{1}{2}z_1(k) + v_1(k), & -\frac{2}{3} \leq v_1(k) \leq \frac{2}{3} \\ z_2(k+1) &= 2z_2(k) + v_2(k), & -\frac{2}{3} \leq v_2(k) \leq \frac{2}{3} \end{aligned}$$

Using the approach of part (a), we get

- for the first component (z_1) that the set of null Ω -controllable values is $S_\infty^{z_1} = (-\infty,\ \infty)$
- for the second component (z_2) that the set of null Ω -controllable values is $S_\infty^{z_2} = (-\frac{2}{3},\ \frac{2}{3})$

We need to express these constraints in terms of the original variables. Since $\mathbf{z} = E^{-1}\mathbf{x}$ we get

$$\begin{aligned} -\infty < z_1 &= \frac{4}{3}x_1 - \frac{2}{3}x_2 < \infty \\ -\frac{2}{3} < z_2 &= -\frac{1}{3}x_1 + \frac{2}{3}x_2 < \frac{2}{3} \end{aligned}$$

The first constraint is always satisfied and so is redundant. The second constraint can be rewritten as

$$-2 < -x_1 + 2x_2 < 2$$

This describes an open set, an infinite strip in the plane between the lines $-2 = -x_1 + 2x_2$ and $2 = -x_1 + 2x_2$. (See Fig. 1.) All points in this strip are null Ω - controllable.

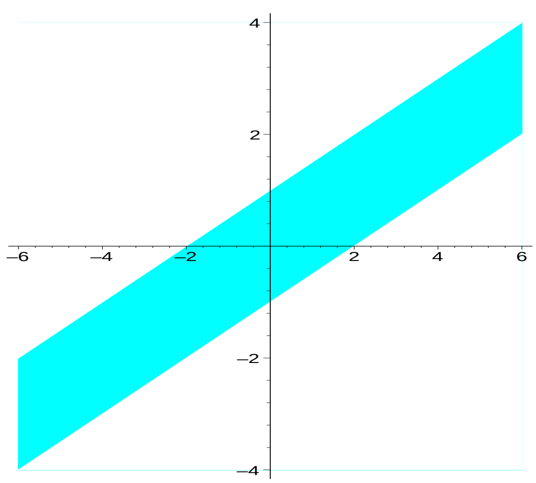

Figure 1: A section of the null controllable states for Q3(c)

If there were no constraints on u, all three would be CC, never mind null controllable.