1. The continuous-time algebraic *Riccati* equation (CARE) applies directly since the system is controllable and with $y = 3x$ is also observable. Here $A = 2$, $B = 1$, $Q = 9$ and $R = 4$. Hence $A^TP + PA - PBR^{-1}B^TP = -Q$ becomes

$$4P - \frac{1}{4}P^2 = -9$$

which has (positive definite) solution $P = 18$. Therefore the feedback controller is given by

$$u = -R^{-1}B^TPx = -\frac{9}{2}x$$

2. Again the CARE applies as the system is controllable and with $y = (\sqrt{8}, 0)\mathbf{x}$ is also observable. Hence $A^TP + PA - PBR^{-1}B^TP = -Q$ becomes

$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} 1^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -p_2 & -p_3 \\ p_1 + 2p_2 & p_2 + 2p_3 \end{pmatrix} + \begin{pmatrix} -p_2 & p_1 + 2p_2 \\ -p_3 & p_2 + 2p_3 \end{pmatrix} - \begin{pmatrix} p_2^2 & p_2p_3 \\ p_2p_3 & p_3^2 \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 0 & 0 \end{pmatrix}$$

or in terms of individual equations

$$\begin{aligned} -2p_2 - p_2^2 &= -8 && \text{position } (1,1) \text{ in matrix equation} \\ p_1 + 2p_2 - p_3 - p_2p_3 &= 0 && \text{position } (1,2) \text{ and/or } (2,1) \text{ in matrix equation} \\ 2p_2 + 4p_3 - p_3^2 &= 0 && \text{position } (2,2) \text{ in matrix equation} \end{aligned}$$

This has the positive definite solution

$$P = \begin{pmatrix} 2 + 6\sqrt{2} & 2 \\ 2 & 2 + 2\sqrt{2} \end{pmatrix}.$$

Hence the feedback controller is

$$u = -R^{-1}B^TP\mathbf{x} = -1^{-1}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 2 + 6\sqrt{2} & 2 \\ 2 & 2 + 2\sqrt{2} \end{pmatrix}\mathbf{x} = -2x_1 - (2 + 2\sqrt{2})x_2.$$

3. The discrete-time matrix *Riccati* equation (DMRE) applies here:
We solve

$$P_k = Q + A^TP_{k+1}A - A^TP_{k+1}B(R + B^TP_{k+1}B)^{-1}B^TP_{k+1}A, \qquad P_N = S$$

for P_k (backwards in time from $k = N$ to $k = 0$), and then we get

$$u_k = -B(R + B^TP_{k+1}B)^{-1}B^TP_{k+1}A\mathbf{x}_k \tag{1}$$

Here

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ R = 1, \ S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

With these values the DMRE becomes

$$\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} p_3' & p_2' \\ p_2' & p_1' \end{pmatrix} - \frac{1}{1 + p_3'}\begin{pmatrix} (p_3')^2 & p_2'p_3' \\ p_2'p_3' & (p_2')^2 \end{pmatrix}, \qquad P_N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

where primed values refer to time $k+1$ and unprimed values to time k. In terms of individual equations, we can write Eq (2) as

$$\begin{aligned} p_1 &= 1 + p_3' - \frac{(p_3')^2}{1+p_3'}, \qquad & p_1(N) = 0 \\ p_2 &= p_2' - \frac{p_2' p_3'}{1+p_3'}, \qquad & p_2(N) = 0 \\ p_3 &= p_1' - \frac{(p_2')^2}{1+p_3'}, \qquad & p_3(N) = 0 \end{aligned} \tag{3}$$

We note that Eq (3) can be solved by "backwards induction" to get

$$p_2 \equiv 0 \tag{4}$$

(try it!).
In terms of the the given values, Eq (1) can be written as

$$\begin{aligned} u &= \frac{-p_3' x_1 - p_2' x_2}{1+p_3'} \\ &= \frac{-p_3'}{1+p_3'} x_1 \qquad \text{using Eq (4)} \\ &= (\alpha_k, 0)\mathbf{x} \end{aligned} \tag{5}$$

where $\alpha_k = \frac{-p_3'}{1+p_3'}$.
In order for a control to exist as $N \to \infty$, it is necessary that the system be stabilisable and in addition it is sufficient that (A, C) be detectable where $Q = C^T C$. Here A, B is completely controllable and with $y = (1,\ 0)\mathbf{x}$ the system is completely observable. Thus $P_k \to P$ where P is the positive definite solution of the discrete-time algebraic *Riccati* equation (DARE):

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

and the optimal u is given by

$$u_k = -B(R + B^T P B)^{-1} B^T P A \mathbf{x}_k$$

In term of the given values we can rewrite the DARE as a system of individual equations

$$p_1 = 1 + p_3 - \frac{(p_3)^2}{1+p_3} \tag{6}$$

$$p_2 = p_2 - \frac{p_2 p_3}{1+p_3} \tag{7}$$

$$p_3 = p_1 - \frac{(p_2)^2}{1+p_3} \tag{8}$$

From Eq (7), (as before) $p_2 = 0$. From Eq (8), this gives $p_3 = p_1$, and so Eq (6) yields $p_3^2 = 1 + p_3$, which has solution $p_3 = \frac{1+\sqrt{5}}{2}$ (chosen to ensure that P is positive definite). Hence $\alpha_k = \frac{p_3}{1+p_3} = \frac{2}{1+\sqrt{5}}$.

4. The HJB equation for this problem is

$$\min_u \{3x^2 + u^2 + \frac{dV}{dx}\left(x^3 + u\right)\} = 0 \tag{9}$$

Differentiating with respect to u gives

$$2u + \frac{dV}{dx} = 0 \qquad \Rightarrow \qquad u = -\frac{1}{2}\frac{dV}{dx}$$

Substituting this into Eq (9) gives

$$\begin{aligned}
3x^2 + x^3\frac{dV}{dx} - \frac{1}{4}\left(\frac{dV}{dx}\right)^2 &= 0 \\
\Rightarrow \frac{dV}{dx} &= 2(x^3 + \sqrt{x^6 + 3x^2}) \qquad \text{chosen to ensure } V(x) \text{ is pd} \\
\Rightarrow u^* &= -x^3 - \sqrt{x^6 + 3x^2} \\
&= -x^3 - x\sqrt{x^4 + 3}
\end{aligned}$$

Substituting this into the state equation gives

$$\dot{x} = -x\sqrt{x^4 + 3}$$

which gives $x(t) \to 0$ as $t \to \infty$.