1. Find the feedback controller that minimises the cost functional

$$J(u) = \int_0^\infty 9x^2 + 4u^2 \, dt$$

for the system

$$\dot{x} = 2x + u.$$

2. Find the feedback controller that minimises the cost functional

$$J(u) = \int_0^\infty 8x_1^2 + u^2 \, dt$$

for the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

3. Show that the feedback controller that minimises

$$J(u) = \sum_{i=0}^{N} x_1(i)^2 + u_i^2$$

for the system

$$\mathbf{x}(k+1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k$$

has the structure

$$u_k = (\alpha_k, 0)\mathbf{x}(k), \qquad 0 \le k < N$$

In the case where $N \to \infty$, show that $\alpha_k \to \frac{2}{1+\sqrt{5}}$

4. Use the *Hamilton-Jacobi-Bellman* equation to find the stabilising feedback controller that minimises the cost functional

$$J(u) = \int_0^\infty 3x^2 + u^2 \, dt$$

for the system

$$\dot{x} = x^3 + u$$