Observability

Completely controllable systems can be restructured by means of state feedback to have many desirable properties. But what if the state is not available for feedback? What if only the output (and input) is available? Observability concerns itself with whether the state can be reconstructed from measurements of output (and input if needed).

1 Continuous-time Systems

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{1}$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \tag{2}$$

DEFINITION 1: The state $\mathbf{x}_0$ is *observable* if there exists an interval $[0,\ t_f)$ such that knowledge of $\mathbf{y}(t)$ (and $\mathbf{u}(t)$) on the interval is sufficient to determine the initial state $\mathbf{x}(0) = \mathbf{x}_0$.
DEFINITION 1A: The system defined by Equations (1) & (2) is completely observable (CO) if every $\mathbf{x}_0$ is observable.

1.1 Linear Time Invariant (LTI) Systems

For the LTI system

$$\dot{\mathbf{x}} = A\,\mathbf{x} + B\,\mathbf{u} \tag{3}$$
$$\mathbf{y} = C\,\mathbf{x} + D\,\mathbf{u} \tag{4}$$

The system of Equations (3) & (4) with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ has output

$$\mathbf{y}(t) = Ce^{At}\mathbf{x}_0 + C\int_0^t e^{-A(t-\tau)}B\mathbf{u}(\tau)d\tau + D\mathbf{u}(t)$$

From which follows

$$\tilde{\mathbf{y}}(t) \stackrel{\triangle}{=} \mathbf{y}(t) - C\int_0^t e^{-A(t-\tau)}B\mathbf{u}(\tau)d\tau - D\mathbf{u}(t) = Ce^{At}\mathbf{x}_0 \tag{5}$$

From knowledge of $\mathbf{y}(t)$ and $\mathbf{u}(t)$, we can compute $\tilde{\mathbf{y}}(t)$ and so observability reduces to determining $\mathbf{x}_0$ from $\tilde{\mathbf{y}}(t)$ via Equation(5). Notice that if there is no input, i.e. $\mathbf{u}(t) \equiv \mathbf{0}$, then $\tilde{\mathbf{y}}(t) = \mathbf{y}(t)$. Thus if there is a method for determining $\mathbf{x}_0$ from $\mathbf{y}(t)$ in the no-input case, then the same method applied to $\tilde{\mathbf{y}}(t)$ in the system-with-input case will also determine $\mathbf{x}_0$.

1.1.1 Modal Criterion

If the state matrix A has distinct eigenvalues, then consider the modal transformation $\mathbf{x} = E\mathbf{z}$ which converts Equations (3) & (4) to

$$\dot{\mathbf{z}} = \Lambda\,\mathbf{z} + \tilde{B}\mathbf{u} \quad (6)$$
$$\mathbf{y} = \tilde{C}\mathbf{z} + D\mathbf{u} \quad (7)$$

Equation (7) may be rewritten

$$\mathbf{y} = \sum_{j=1}^{n} \tilde{C}_j z_j + D\mathbf{u}$$

where $\tilde{C}_j$ is the j-th column of $\tilde{C}$.
Thus, a necessary condition for $\mathbf{y}$ to contain information on the mode z_j is that $\tilde{C}_j \neq \mathbf{0}$, or, more generally, a necessary condition for Equations (6) & (7) to be CO is that no column of $\tilde{C}$ be zero. This is also sufficient. This leads to

MODAL CRITERION: Equations (6) & (7) is CO if and only if no column of $\tilde{C}$ is zero.

We can categorise the j-th mode of a system as observable or unobservable depending on whether it can be computed from $\mathbf{y}$ or not. The system will be practically useless if any unobservable modes are unstable. To highlight this, we say

DEFINITION 2: The system of Equations (3) & (4)) is *detectable* if all its unobservable modes are asymptotically stable.

The effects of controllability and observability on the modes of a system are contained in the *Kalman* Decomposition or Canonical Structure Theorem which states that each mode of a system falls in one of the four categories:

- both controllable and observable
- controllable but not observable
- observable but not controllable
- neither controllable nor observable.

The transfer function of a linear system relates how the output depends on the input. Hence only modes in the first category appear in the transfer function description of a system.

1.1.2 Rank Criterion

Repeatedly differentiating Equation (5) with respect to time gives
$\frac{d^i\tilde{\mathbf{y}}}{dt^i} = CA^i e^{At}\mathbf{x}_0$ for $i \geq 0$: [In order to obtain these derivatives it is necessary to have measurements of $\mathbf{y}(t)$ over an interval]. Evaluating each of these at $t = 0$ gives $\frac{d^i\tilde{\mathbf{y}}}{dt^i}(0) = CA^i\mathbf{x}_0$. For $i = 0, 1, \ldots, n-1$ this in turn gives

$$M(\tilde{y}) = V\mathbf{x}_0 \quad (8)$$

where

$$M(\tilde{y}) \triangleq \begin{pmatrix} \tilde{\mathbf{y}}(0) \\ \frac{d\tilde{\mathbf{y}}}{dt}(0) \\ \frac{d^2\tilde{\mathbf{y}}}{dt^2}(0) \\ \vdots \\ \frac{d^{n-1}\tilde{\mathbf{y}}}{dt^{n-1}}(0) \end{pmatrix}, \qquad V \triangleq \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

Given the measurements $M(\tilde{y})$, when can Equation (8) be solved for $\mathbf{x}_0$? Equation (8) can be solved given arbitrary $M(\tilde{y})$ if and only if V has full rank. Since V is $nm \times n$, this leads to

RANK CRITERION: The system of Equations (3) & (4) is CO if and only if $\text{rank}(V) = n$.

We also say that the pair (A, C) is CO whenever V has full rank.

1.1.3 ***Popov-Belevitch-Hautus*** (PBH) Test for Observability

Consider the $(m + n) \times n$ matrix

$$N \triangleq \begin{bmatrix} C \\ \lambda I - A \end{bmatrix}, \qquad \text{for all } \lambda \in \mathbf{C}. \tag{9}$$

TEST: The system of Eqn (2) is CO if and only if

$$\text{rank } N = n.$$

We note that if λ is not an eigenvalue of A, then $\lambda I - A$ has rank n and so the test is automatically satisfied. Furthermore this means we only need to check the rank of N at the eigenvalues of A. If the test fails at a particular value of λ, then the mode corresponding to that eigenvalue is unobservable.

In addition we can adapt the PBH Test to check for detectability. Recall that a system is detectable if and only if its unstable modes are observable. Thus by performing the PBH test for the unstable eigenvalues of A will establish whether the system is detectable or not.

1.2 LTI systems: Observable Canonical Form (OCF)

We shall look at one which is used in several design algorithms, but we shall restrict our attention to single output systems. For the purposes of this section, we have set $\mathbf{u}(t) \equiv 0$.

Consider an arbitrary single output system

$$\dot{\mathbf{x}} = A\,\mathbf{x} \qquad y = c\mathbf{x} \tag{10}$$

with observability matrix

$$V = \begin{pmatrix} c \\ cA \\ cA^2 \\ \cdots \\ cA^{n-1} \end{pmatrix}$$

The single output system

$$\dot{\hat{x}} = \hat{A}\hat{\mathbf{x}} \qquad \hat{y} = \hat{c}\hat{\mathbf{x}} \tag{11}$$

where

$$\hat{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}, \qquad \hat{c} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{12}$$

has observability matrix

$$\hat{V} = \begin{pmatrix} \hat{c} \\ \hat{c}\hat{A} \\ \hat{c}\hat{A}^2 \\ \cdots \\ \hat{c}\hat{A}^{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & - \\ \vdots & \vdots & & - & - \\ 0 & 1 & \cdots & - & - \\ 1 & - & \cdots & - & - \end{pmatrix}$$

Where the $-$ indicate an irrelevant entry. Note that $\hat{A}$ is the transpose of a companion form matrix. The system described by Equation (11) is said to be in *observable canonical form* (OCF).

The rank of $(\hat{U})$ is n and so the system is CO.

We can transform the arbitrary system Equation (10) into OCF if and only if the pair (A, c) is CO. More specifically, this is achieved by the transformation $\mathbf{x} = \hat{T}\hat{\mathbf{x}}$ where $\hat{T} = (\hat{V}^{-1}V)^T$. Exercise: Prove this result.

2 Discrete-Time Systems

Most of the definitions and concepts associated with continuous-time systems carry over to discrete-time with only the obvious modifications.
For the system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) \tag{13}$$
$$\mathbf{y}_k = \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k) \tag{14}$$

DEFINITION 1: The state $\mathbf{x}_0$ is *observable* if there exists output (and input) sequences of length N such that knowledge of $\mathbf{y}_k$ (and $\mathbf{u}_k$) on these sequences is sufficient to determine the initial state $\mathbf{x}_0$.

Definition 1A is unchanged.

2.1 LTI Systems

The system is

$$\mathbf{x}_{k+1} = A\,\mathbf{x}_k + B\,\mathbf{u}_k \tag{15}$$
$$\mathbf{y}_k = C\,\mathbf{x}_k + D\,\mathbf{u}_k \tag{16}$$

leading to an equivalent expression to Equation (5):

$$\tilde{\mathbf{y}}_k \triangleq \mathbf{y}_k - C \sum_{j=0}^{k-1} A^{k-1-j} B u_j - D u_k = C A^k \mathbf{x}_0 \tag{17}$$

2.2 Observability Criteria and Canonical Forms

The Modal and Rank criteria and PBH test carry over directly from the continuous-time case along with the accompanying concepts of observable modes and states and OCF. The only new point of interest is in the interpretation of Equation (8) in the discrete-time case. Here it is

$$M(\tilde{\mathbf{y}}) = V \mathbf{x}_0$$

where

$$M(\tilde{\mathbf{y}}) \triangleq \begin{pmatrix} \tilde{\mathbf{y}}_0 \\ \tilde{\mathbf{y}}_1 \\ \tilde{\mathbf{y}}_2 \\ \vdots \\ \tilde{\mathbf{y}}_{n-1} \end{pmatrix}$$

3 Duality

Let A, B, and C be given $n \times n$, $n \times r$ and $m \times n$ matrices respectively with real entries. Consider the LTI systems

$$S_1 : \qquad \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad\qquad y = C\mathbf{x}$$

and

$$S_2 : \qquad \dot{\mathbf{z}} = A^T\mathbf{z} + C^T\mathbf{v}, \qquad\qquad \mathbf{w} = B^T\mathbf{z}$$

They are said to be dual systems of each other. Due to their definitions, they have the properties that

- S_1 is CC if and only if S_2 is CO.
  To see why this is so, consider the controllability matrix for S_1 which is $U = [B, AB, A^2B, \ldots A^{n-1}B]$. The observability matrix for S_2 is (when transposed) $V^T = [B, AB, A^2B, \ldots A^{n-1}B] = U$.

- S_1 is CO if and only if S_2 is CC.

These properties are useful in that

- a result derived on the basis of the CC of S_1 applies to S_2 if it is CO, and vice versa

- a result derived on the basis of the CO of S_1 applies to S_2 if it is CC, and vice versa.

Similar ideas hold for discrete-time LTI dual systems.