Periodic Orbits of Discrete-Time systems

The autonomous discrete-time time-invariant system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

has a periodic orbit : $\mathbf{z}_1, \ \mathbf{z}_2, \ldots, \mathbf{z}_p$ (of period p) if

$$\begin{aligned} \mathbf{z}_{i+1} &= \mathbf{f}(\mathbf{z}_i) \qquad && i = 1, 2, \ldots, p-1 \\ \mathbf{z}_1 &= \mathbf{f}(\mathbf{z}_p) && \\ \mathbf{z}_1 &\neq \mathbf{f}(\mathbf{z}_i) && i = 1, 2, \ldots, p-1 \end{aligned}$$

or, alternatively,

$$\begin{aligned} \mathbf{z}_i &= \mathbf{f}^{(p)}(\mathbf{z}_i) \qquad && i = 1, 2, \ldots, p && (1) \\ \mathbf{z}_i &\neq \mathbf{f}^{(j)}(\mathbf{z}_i) && i = 1, 2, \ldots, p, \quad 1 \leq j < p && (2) \end{aligned}$$

where

$$\mathbf{f}^{(j)} = \underbrace{\mathbf{f} \circ \mathbf{f} \circ \cdots \circ \mathbf{f}}_{j \ \text{times}}$$

Thus, each point of the orbit can be regarded as a fixed point of the system

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k) \qquad \text{where} \quad \mathbf{F} = \mathbf{f}^{(p)} \tag{3}$$

To compute period-p orbits of $\mathbf{f}$, it suffices to find all fixed points of Eq(3), and then eliminate those that correspond to lower period orbits of $\mathbf{f}$. (What are the lower period orbits of the states that are eliminated ?)

The local stability of the fixed points of Eq(3) may be investigated by looking at the eigenvalues of the linearised map:

$$\delta\mathbf{x}_{k+1} = A\delta\mathbf{x}_k \qquad \text{where} \quad A = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}$$

It is interesting to note that A may be written as

$$A = M_1 \stackrel{def}{=} A_p \, A_{p-1} \cdots A_2 \, A_1 \tag{4}$$

where

$$A_i = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{z}_i) \qquad i = 1, 2, \ldots, p \tag{5}$$

Exercise: Prove that Eq(4) holds for the case of a 2nd order system with period 2.

Eq(4) was derived by linearising the periodic orbit starting in state $\mathbf{z}_1$. But any of the other states of the periodic orbit could have been used just as well. For instance if the orbit started in state $\mathbf{z}_2$, then A would have been computed as

$$A = M_2 \stackrel{def}{=} A_1\, A_p\, A_{p-1} \cdots A_3\, A_2 \tag{6}$$

What happens to the eigenvalues in that case ? Consider the following:
Let λ and $\mathbf{e}_1$ be an eigenvalue and corresponding eigenvector of M_1, then

$$\begin{aligned} M_1\, \mathbf{e}_1 &= \lambda\, \mathbf{e}_1 \\ \Rightarrow \quad A_1\, (M_1\, \mathbf{e}_1) &= A_1\, (\lambda\, \mathbf{e}_1) \\ \Rightarrow \quad A_1\, (A_p\, A_{p-1} \cdots A_2\, A_1)\, \mathbf{e}_1 &= \lambda\, A_1\, \mathbf{e}_1 \\ \Rightarrow \quad (A_1 A_p\, A_{p-1} \cdots A_2)\, A_1\, \mathbf{e}_1 &= \lambda\, A_1\, \mathbf{e}_1 \\ \Rightarrow \quad M_2\, (A_1\, \mathbf{e}_1) &= \lambda\, (A_1\, \mathbf{e}_1) \end{aligned}$$

Thus M_1 and M_2 have the same eigenvalues, but different corresponding eigenvectors.