Piecewise Linear Maps on an Interval

Piecewise linear maps are frequently met as approximations to nonlinear systems or as exact models of man-made or engineered systems.
We have already met the tent map (see Problem Sheet 2 - Q2).
As another example, consider the piecewise linear dynamical system defined on the interval $[0, 2]$ by the map

$$x' = f(x) = \begin{cases} \frac{3}{2}x, & \text{if } 0 \leq x < 1 \\ x + \frac{1}{2}, & \text{if } 1 \leq x < 3/2 \\ 8 - 4x, & \text{if } 3/2 \leq x \leq 2 \end{cases}$$

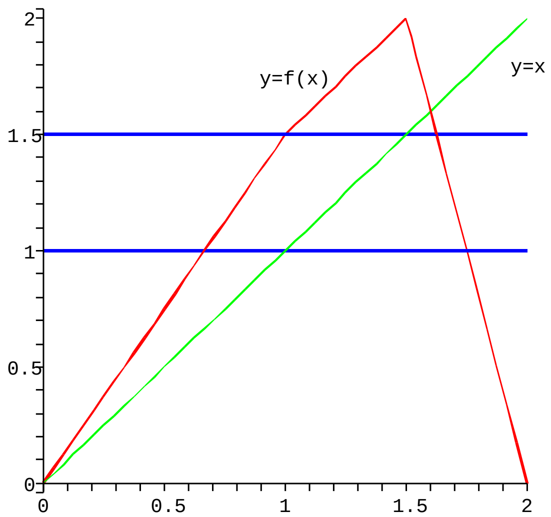

Figure 1: $f(x)$

As usual, the fixed points are given by

$$x_e = f(x_e) = \begin{cases} \frac{3}{2}x_e, & \text{if } 0 \leq x_e < 1 \\ x_e + \frac{1}{2}, & \text{if } 1 \leq x_e < 3/2 \\ 8 - 4x_e, & \text{if } 3/2 \leq x_e \leq 2 \end{cases}$$

$$\Rightarrow \begin{cases} x_e = 0, & \text{if } 0 \leq x_e < 1 \\ x_e = \text{ not defined}, & \text{if } 1 \leq x_e < 3/2 \\ x_e = \frac{8}{5}, & \text{if } 3/2 \leq x_e \leq 2 \end{cases}$$

From Fig. 1, both fixed points are unstable. Indeed for small $\delta > 0$, for $x_e = 0$

$$x = 0 + \delta \Rightarrow x' = \frac{3}{2}\delta \Rightarrow |x' - 0| = \frac{3}{2}\delta > \delta = |x - 0|$$

and for $x_e = 8/5$

$$x = \frac{8}{5} \pm \delta \Rightarrow x' = 8 - 4(\frac{8}{5} \pm \delta) \Rightarrow |x' - \frac{8}{5}| = 4\delta > \delta = |x - \frac{8}{5}|$$

Alternatively for $x_e = 8/5$

$$f'\left(\frac{8}{5}\right) = -4$$

Period-2 points are prime solutions of

$$f^2(x) = x$$

From the definition of the map

$$f^2(x) = \begin{cases} \frac{3}{2}f(x), & \text{if } 0 \le f(x) < 1 \\ f(x) + \frac{1}{2}, & \text{if } 1 \le f(x) < 3/2 \\ 8 - 4f(x), & \text{if } 3/2 \le f(x) \le 2 \end{cases}$$

and using the blue lines in Fig. 1 this becomes

$$f^2(x) = \begin{cases} \frac{3}{2}f(x), & \text{if } 0 \le x < 2/3 \text{ or } 7/4 \le x \le 2 \\ f(x) + \frac{1}{2}, & \text{if } 2/3 \le x < 1 \text{ or } 13/8 \le x < 7/4 \\ 8 - 4f(x), & \text{if } 1 \le x < 3/2 \text{ or } 3/2 \le x < 13/8 \end{cases}$$

which "unravels" to

$$f^2(x) = \begin{cases} (\frac{3}{2})^2 x = \frac{9}{4}x, & \text{if } 0 \le x < 2/3 \\ \frac{3}{2}x + \frac{1}{2}, & \text{if } 2/3 \le x < 1 \\ 8 - 4(x + \frac{1}{2}) = 6 - 4x, & \text{if } 1 \le x < 3/2 \\ 8 - 4(8 - 4x) = 16x - 24, & \text{if } 3/2 \le x < 13/8 \\ 8 - 4x + \frac{1}{2} = \frac{17}{2} - 4x, & \text{if } 13/8 \le x < 7/4 \\ \frac{3}{2}(8 - 4x) = 12 - 6x, & \text{if } 7/4 \le x \le 2 \end{cases}$$

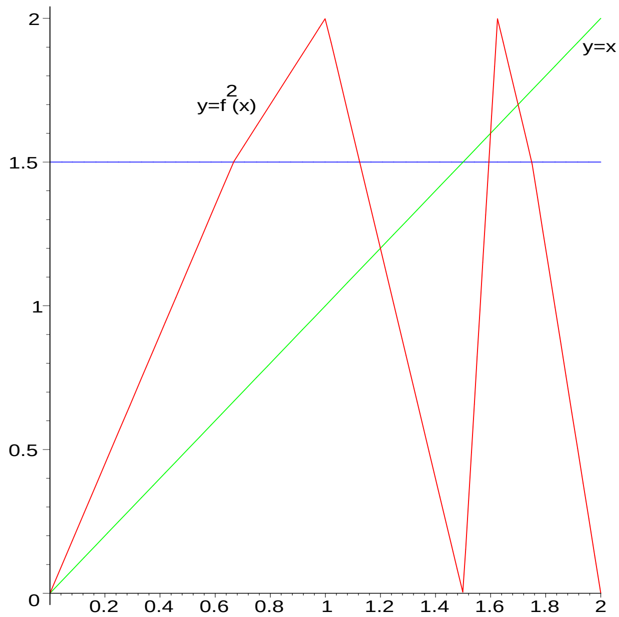

Figure 2: $f^2(x)$

Thus

$$x = f^2(x) = \begin{cases} (9/4)x, & \text{if } 0 \le x < 2/3 \\ (3/2)x + 1/2, & \text{if } 2/3 \le x < 1 \\ 6 - 4x, & \text{if } 1 \le x < 3/2 \\ 16x - 24, & \text{if } 3/2 \le x < 13/8 \\ 17/2 - 4x, & \text{if } 13/8 \le x < 7/4 \\ 12 - 6x, & \text{if } 7/4 \le x \le 2 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0, & \text{if } 0 \le x < 2/3\text{: fixed point} \\ x = -1/2, & \text{if } 2/3 \le x < 1\text{: not in subinterval} \\ x = 6/5, & \text{if } 1 \le x < 3/2 \\ x = 24/15 = 8/5, & \text{if } 3/2 \le x < 13/8\text{: fixed point} \\ x = 17/10, & \text{if } 13/8 \le x < 7/4 \\ x = 12/7, & \text{if } 7/4 \le x \le 2\text{: not in subinterval} \end{cases}$$

Thus the period-2 orbit is potentially $\{6/5,\ 17/10\}$. Checking this we get

$$f\left(\frac{6}{5}\right) = \frac{6}{5} + \frac{1}{2} = \frac{17}{10}, \qquad f\left(\frac{17}{10}\right) = 8 - 4\left(\frac{17}{10}\right) = \frac{6}{5}$$

The orbit is unstable since

$$f'\left(\frac{6}{5}\right) \times f'\left(\frac{17}{10}\right) = (-4) \times (-4)$$

(see Fig.2)

In a similar fashion, we can compute period-3 points. However, using a tool like Maple, makes this a lot easier.

Exercise: Find the fixed points and period-2 points of

$$x' = f(x) = \begin{cases} 2x, & \text{if } 0 \le x < 1 \\ 3 - x, & \text{if } 1 \le x \le 2 \end{cases}$$

Use Maple to answer: Does this map have period-3 points ?