

1. Linear State Equations

- (a) The LTI discrete-time equation $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, $\mathbf{x}_{k_0} = \mathbf{x}_0$ has solution

$$\mathbf{x}_k = A^{k-k_0}\mathbf{x}_0 + \sum_{i=k_0}^{k-1} A^{k-1-i}B\mathbf{u}_i$$

- (b) The LTI continuous-time equation $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, $\mathbf{x}(t_0) = \mathbf{x}_0$ has solution

$$\mathbf{x}(t) = e^{A(t-t_0)}\{\mathbf{x}_0 + \int_{t_0}^t e^{A(t_0-\tau)}B\mathbf{u}(\tau) d\tau\}$$

2. Computing A^k and e^{At} when A is diagonalisable ($A = E\Lambda E^{-1}$)

$$A^k = E\Lambda^k E^{-1}$$

$$e^{At} = E \left(\sum_{j=0}^{\infty} \frac{\Lambda^j t^j}{j!} \right) E^{-1} = E e^{\Lambda t} E^{-1}$$

3. *Lyapunov's* Direct Method for Linear systems

- (a) The (continuous-time) *Lyapunov* Equation is

$$A^T P + P A = -Q$$

- (b) The Discrete *Lyapunov* Equation is

$$A^T P A - P = -Q$$

4. Transformations

The transformation $\mathbf{x} = T\mathbf{z}$ transforms $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, $\mathbf{y} = C\mathbf{x}$ to $\dot{\mathbf{z}} = T^{-1}AT\mathbf{z} + T^{-1}B\mathbf{u}$, $\mathbf{y} = C T\mathbf{x}$.

5. Controllability, Observability, & Duality

- (a) The Controllability U and Observability V matrices are given by:

$$U = (B, AB, A^2B, \dots, A^{n-1}B) \quad V = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

- (b) *Popov-Belevitch-Hautus* (PBH) test for controllability:

$$M \triangleq [\lambda I - A, B]$$

is of rank n for all λ if and only if (A, B) is completely controllable.

(c) The PBH test for observability:

$$N \triangleq \begin{bmatrix} C \\ \lambda I - A \end{bmatrix}$$

is of rank n for all λ if and only if (A, C) is completely observable.

(d) Controllable Canonical Form for a Single Input System with characteristic polynomial $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

(e) The system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, $\mathbf{y} = C\mathbf{x}$
has dual $\dot{\mathbf{x}}^* = A^T\mathbf{x}^* + C^T\mathbf{u}^*$, $\mathbf{y}^* = B^T\mathbf{x}^*$

6. Pole Placement for Single Input Systems

The transformation $\hat{T} = U\hat{U}^{-1}$, where U and \hat{U} are the controllability matrices of the original and transformed systems respectively, transforms a completely controllable system into controllable canonical form.

The matrix $K = \hat{K}\hat{T}^{-1}$ where $\hat{k}_i = a_{i-1} - \hat{a}_{i-1}$, $i = 1, 2, \dots, n$ is the required feedback matrix.

7. Luenberger Observers

The continuous-time full order *Luenberger* observer is

$$\dot{\mathbf{z}} = (A + LC)\mathbf{z} + B\mathbf{u} - L\mathbf{y}$$

8. Decoupling

Let d_i be the smallest nonnegative integer such that $c_i A^{d_i} B \neq 0$. Then

$$\hat{\mathbf{y}} = \begin{pmatrix} c_1 A^{d_1+1} \\ c_2 A^{d_2+1} \\ \vdots \\ c_m A^{d_m+1} \end{pmatrix} \mathbf{x} + \begin{pmatrix} c_1 A^{d_1} B \\ c_2 A^{d_2} B \\ \vdots \\ c_m A^{d_m} B \end{pmatrix} \mathbf{u}$$

where $\hat{y}_i = \frac{d^{d_i+1} y_i}{dt^{d_i+1}}$ for continuous-time or $\hat{y}_i = y_i(k + d_i + 1)$ for discrete-time systems.

The *remnant*, if it exists, is of order $q = n - \sum (d_i + 1)$.

9. Hamilton-Jacobi-Bellman (HJB) Equation for infinite horizon problems.

The stabilising control that minimises $J = \int_{t_0}^{\infty} L(\mathbf{x}, \mathbf{u}) dt$ for the system

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, $\mathbf{x}(t_0) = \mathbf{x}_0$ is given by the solution of

$$0 = \min_{\mathbf{u}(t)} \left\{ L(\mathbf{x}, \mathbf{u}) + \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T \mathbf{f}(\mathbf{x}, \mathbf{u}) \right\}$$

where $V(\mathbf{x}) = \min_{\mathbf{u}(\cdot)} J$.

10. Linear Quadratic (LQ) control

The minimum value of

$$J(\mathbf{u}) = \mathbf{x}^T(t_f)S\mathbf{x}(t_f) + \int_{t_0}^{t_f} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} dt$$

for the system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by $V(\mathbf{x}_0, t_0)$ where $V(\mathbf{x}, t) = \mathbf{x}^T P \mathbf{x}$ and $P = P^T$ is the solution of the matrix *Riccati* equation (MRE)

$$-\dot{P} = Q + A^T P + P A - P B R^{-1} B^T P \quad P(t_f) = S$$

Furthermore $\mathbf{u}(t) = -R^{-1} B^T P \mathbf{x}(t)$.

Additionally, if $t_f \rightarrow \infty$, then for the LTI system with (A, B) completely controllable and (A, C) completely observable where $C^T C = Q$, P is the positive definite solution of the continuous-time algebraic *Riccati* equation (CARE)

$$0 = Q + A^T P + P A - P B R^{-1} B^T P$$

11. *LaSalle* Invariance Principle: For the system

$$\dot{x} = \mathbf{f}(\mathbf{x}), \quad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

if (i) V is positive definite, (ii) $\frac{dV}{dt}$ is negative semi-definite and (iii) the union of all invariant sets with $\frac{dV}{dt} = 0$ is $M = \{\mathbf{0}\}$, then $\mathbf{0}$ is asymptotically stable. In particular the system

$$\ddot{x} + b(\dot{x}) + c(x) = 0,$$

where b and c are continuous functions with the same sign as their arguments, has $\mathbf{0}$ asymptotically stable.

12. Control *Lyapunov* Functions

$V(\mathbf{x})$ is a control *Lyapunov* function for the affine control system

$$\dot{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \quad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

if (i) it is positive definite and (ii)

$$b(\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad a(\mathbf{x}) < 0, \quad \mathbf{x} \neq \mathbf{0}$$

where

$$\frac{dV}{dt} = \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T \dot{\mathbf{x}} = \underbrace{\left(\frac{\partial V}{\partial \mathbf{x}} \right)^T \mathbf{f}(\mathbf{x})}_{a(\mathbf{x})} + \underbrace{\left(\frac{\partial V}{\partial \mathbf{x}} \right)^T \mathbf{g}(\mathbf{x}) \mathbf{u}}_{b(\mathbf{x})}$$

Sontag's formula: for a single input system a stabilising control is

$$u = k(\mathbf{x}) \triangleq \begin{cases} -\frac{a(\mathbf{x}) + \sqrt{a^2(\mathbf{x}) + b^4(\mathbf{x})}}{b(\mathbf{x})}, & \text{if } b(\mathbf{x}) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

13. Differential Geometry notations:

$$L_{\mathbf{f}}h = \left[\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \dots, \frac{\partial h}{\partial x_n} \right] \mathbf{f}$$

$$ad_{\mathbf{f}} \mathbf{g} = [\mathbf{f}, \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \mathbf{f} - \frac{\partial \mathbf{f}}{\partial v x} \mathbf{g}$$

$$ad_{\mathbf{f}^0} \mathbf{g} = \mathbf{g}, \quad ad_{\mathbf{f}^{k+1}} \mathbf{g} = [\mathbf{f}, ad_{\mathbf{f}^k} \mathbf{g}], \quad k = 0, 1, \dots$$

14. Feedback Linearisation

For a single input affine control system, the linearising transformation and linearising control are

$$\mathbf{z} = \Phi(\mathbf{x}), \quad u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v$$

Algorithm:

(a) compute the set of vector fields

$$F_n = \{ \mathbf{g}, ad_{\mathbf{f}} \mathbf{g}, ad_{\mathbf{f}^2} \mathbf{g}, \dots, ad_{\mathbf{f}^{n-1}} \mathbf{g} \}$$

(b) check that there exists a region \mathcal{N} such that (i) F_n is linearly independent (equivalent to the system being controllable) and (ii) F_{n-1} is involutive (meaning that a certain set of pdes can be solved). If so then...

(c) Solve the system of pdes

$$\begin{aligned} L_{ad_{\mathbf{f}^i} \mathbf{g}} z_1 &= 0, & i &= 0, 1, \dots, n-2 \\ L_{ad_{\mathbf{f}^{n-1}} \mathbf{g}} z_1 &\neq 0 \end{aligned}$$

for z_1 the first component of \mathbf{z} .

(d) Compute the state transformation

$$\mathbf{z}(\mathbf{x}) = \Phi(\mathbf{x}) = (z_1, L_{\mathbf{f}} z_1, L_{\mathbf{f}^2} z_1, \dots, L_{\mathbf{f}^{n-1}} z_1)$$

and the functions

$$\beta(\mathbf{x}) = \frac{1}{L_{\mathbf{g}} L_{\mathbf{f}^{n-1}} z_1}, \quad \alpha(\mathbf{x}) = -\frac{L_{\mathbf{f}^n} z_1}{L_{\mathbf{g}} L_{\mathbf{f}^{n-1}} z_1}.$$