Vector Calculus

February 24, 2017

Chapter 1

Vector functions of a real variable

1.1 Definition of a Vector function

Definition 1.1.1 *Suppose the components of a vector*

$$\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t)), \tag{1.1}$$

are single-valued functions of a real variable. Then $\mathbf{f}$ *is called a* **vector function** *of* t.

$\mathbf{f}(t)$ is a continuous function of t if $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous functions. (Roughly speaking, a function is continuous if its value does not change suddenly at any point).

Examples:

$$\begin{aligned} \mathbf{f}(t) &= (2, \sqrt{t}, \sin t), \qquad 0 \le t < \infty \\ \mathbf{f}(t) &= (t^3, t, 3), \qquad -\infty \le t \le 2 \\ \mathbf{f}(t) &= (2t^2, 2, 6t^{-1}), \qquad 2 < t < \infty. \end{aligned}$$

1.2 Geometrical Representation of a Vector function

Consider a position vector $\overrightarrow{OP}$ where O is the origin and P is the point $\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$. As t varies over it's range of values, P describes a curve in 3 dimensions. The equation

$$\overrightarrow{OP} = \mathbf{r} = \mathbf{f}(t), \tag{1.2}$$

where $\mathbf{r} = (x, y, z)$ is called the **parametric equation** of the curve described by P (t is the parameter). Here t is called the parameter and to completely specify the curve, the range over which t varies must also be given as in the examples above (see Figure 1.1).

Example 1.2.1 *Find the locus of* P *as* θ *varies* $(0 \le \theta \le 2\pi)$ *(with* α *constant) if*

$$\overrightarrow{OP} = (\alpha \cos\theta, 0, \alpha \sin\theta).$$

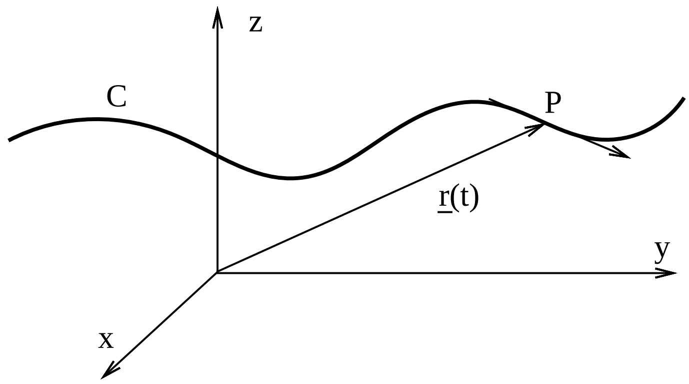

Figure 1.1: *A curve in 3D described by a point P whose position is given by an equation of the type* $\mathbf{r} = \mathbf{f}(t)$.

Example 1.2.2 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be the position vectors relative to the origin of the points A, B. Show that the equation of the straight line through A, B can be expressed in the form:*

$$\mathbf{r} = \mathbf{a} + (\mathbf{b} - \mathbf{a})t, \tag{1.3}$$

where t is a parameter in the range $-\infty < t < \infty$.

Solution. The position vector of B relative to A is

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

The point P with position vector $\mathbf{r}$ lies on the line through A and B (see Figure 1.2) if and only if

$$\overrightarrow{AP} = (\mathbf{b} - \mathbf{a})t,$$

where t is some real number. Noting that

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP},$$

we have $\mathbf{r} = \mathbf{a} + (\mathbf{b} - \mathbf{a})t$. This is the parametric equation of the straight line through A and B because the position vector of all points on the line can be represented in this form.
Intuitively note that the vector $\mathbf{b} - \mathbf{a}$ is parallel to $\overrightarrow{AB}$ so in equation (1.3) the first term on the RHS picks out the point A and the second term moves the point in a direction parallel to the line AB. The value of t determines which particular point on the line is picked out.

1.3 Differentiation of Vectors

Definition 1.3.1 *Suppose* $\mathbf{f}(t) = (f_1(t), f_2(t), f_3(t))$ *and* $f_i(t)$ *are differentiable with t in some given interval. Then we define*

$$\frac{d\mathbf{f}}{dt} = \left(\frac{df_1}{dt}, \frac{df_2}{dt}, \frac{df_3}{dt}\right), \tag{1.4}$$

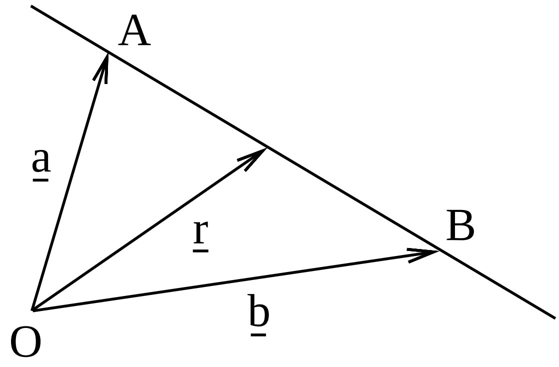

Figure 1.2: *Equation of the straight line through A, B.*

to be the **first derivative** *of* $\mathbf{f}(t)$.

There is a natural extension to higher derivatives $\frac{d^n\mathbf{f}}{dt^n}$, e.g.

$$\frac{d^2\mathbf{f}}{dt^2} = \left(\frac{d^2 f_1}{dt^2}, \frac{d^2 f_2}{dt^2}, \frac{d^2 f_3}{dt^2}\right).$$

Example 1.3.1 *Find the values for which* $\mathbf{a} = (\cos\lambda x, \sin\lambda x, 0)$ *satisfies the differential equation*

$$\frac{d^2\mathbf{a}}{dx^2} = -9\mathbf{a}.$$

1.4 Differentiation rules

If $\mathbf{a}(t)$, $\mathbf{b}(t)$ and $\lambda(t)$ (a scalar) are differentiable w.r.t t:

$$\begin{aligned}
&\text{(i)} && \frac{d(\mathbf{a}+\mathbf{b})}{dt} = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt} \\
&\text{(ii)} && \frac{d(\lambda\mathbf{a})}{dt} = \lambda\frac{d\mathbf{a}}{dt} + \frac{d\lambda}{dt}\mathbf{a} \\
&\text{(iii)} && \frac{d(\mathbf{a}\cdot\mathbf{b})}{dt} = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \frac{d\mathbf{b}}{dt}\cdot\mathbf{a} \\
&\text{(iv)} && \frac{d(\mathbf{a}\times\mathbf{b})}{dt} = \frac{d\mathbf{a}}{dt}\times\mathbf{b} + \mathbf{a}\times\frac{d\mathbf{b}}{dt}.
\end{aligned}$$

Note: The order is important in (iv) but not in (iii). The operation of taking the dot product of two vectors is commutative; the operation of taking the vector product is not. I.e. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ but $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$.

Example 1.4.1 *Show that the first derivative of a unit vector* $\hat{\mathbf{a}} = \hat{\mathbf{a}}(t)$ *is always perpendicular to* $\hat{\mathbf{a}}$ *provided the derivative is not zero.*

Solution.

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = 1 \quad \Rightarrow \quad \frac{d\hat{\mathbf{a}}}{dt} \cdot \hat{\mathbf{a}} + \hat{\mathbf{a}} \cdot \frac{d\hat{\mathbf{a}}}{dt} = 0 \quad \Rightarrow \quad 2\hat{\mathbf{a}} \cdot \frac{d\hat{\mathbf{a}}}{dt} = 0,$$

which implies that $\hat{\mathbf{a}}$ is perpendicular to $\frac{d\hat{\mathbf{a}}}{dt}$.

Example 1.4.2 *Apply the above to the particular example where* $\hat{\mathbf{a}} = (\cos t, \sin t, 0)$.

1.5 The tangent to a curve

Suppose a continuous curve C (i.e. a curve without any break or jump, which can be drawn without removing the pen from the paper) is the locus of the point P whose position vector relative to the origin O is described by

$$\overrightarrow{OP} = \mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t)). \tag{1.5}$$

(Note that we could have written $\mathbf{r} = \mathbf{f}(t)$ but the common practice is to use $\mathbf{r}$ to symbolise the function). Let P be a particular point on C at which $\frac{d\mathbf{r}}{dt}$ exists and is not zero.
Then at this point $\frac{d\mathbf{r}}{dt}$ lies along the tangent to the curve in the sense in which the curve is described by P as t increases (see Figure 1.3).

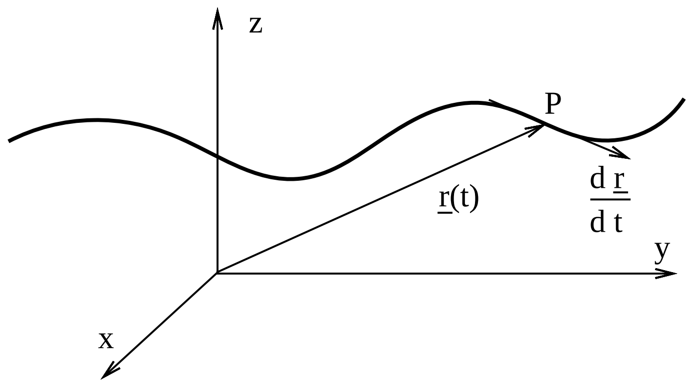

Figure 1.3: *Tangent to a curve:* $\frac{d\mathbf{r}}{dt}$ *is tangent at the point* P_0.

If the tangent at P is $\frac{d\mathbf{r}}{dt}$ at $t = t_0$, then the **unit tangent** is defined to be the vector:

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}. \tag{1.6}$$

Example 1.5.1 *Consider the vector valued function* $\mathbf{r}(t) = (t, t^3, 1)$. *Find its derivative and hence the unit tangent vector to the curve at the point* $(0, 0, 1)$.

1.6 Smoothness

The curve described by $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t))$, is said to be **smooth** if $\hat{\mathbf{t}}$ exists at all points and is continuous. Smoothness means the curve does not undergo any sudden changes in **direction**. (See Figure 1.4).

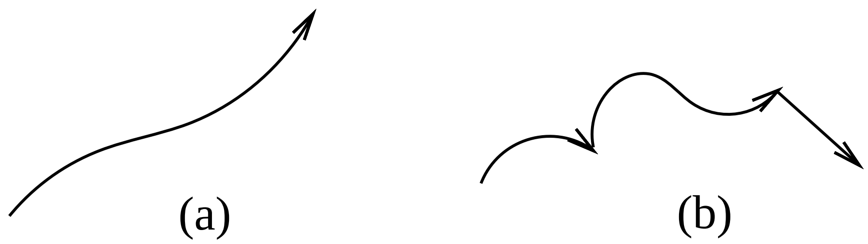

Figure 1.4: *Classification of curves: (a) is smooth and (b) is piecewise smooth.*

A **piecewise smooth** curve $\mathbf{r} = \mathbf{r}(t)$ is one which is continuous and consists of a finite number of smooth curves linked end to end.

Example 1.6.1 *Show that the unit tangent to the curve*

$$\mathbf{r}(t) = \begin{cases} (t^2, 2t, 0), & -1 \le t \le 1 \\ (1, 4-2t, 0), & 1 < t \le 2, \end{cases}$$

is discontinuous at $t = 1$. Verify that it is piecewise smooth.

1.7 Arclength

Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a parametric equation of a **piecewise** smooth curve. Define

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}, \tag{1.7}$$

and so,

$$s(t) = \int_{t_0}^{t} \left|\frac{d\mathbf{r}}{dt}\right| dt = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt, \tag{1.8}$$

is the **arclength** of C from the fixed point t_0 to the variable point t. Suppose the curve extends from A to B with $\mathbf{r}(t_0) = \overrightarrow{OA}$ and $\mathbf{r}(t_1) = \overrightarrow{OB}$. Then $s(t_0) = 0$ and $s(t_1)$ is the length of curve from A to B.

The element of arclength ds satisfies $ds^2 = dx^2 + dy^2 + dz^2$ and is a natural extension of the 2D situation (see Figure 1.5).

Example 1.7.1 *Find an expression for the arclength of the curve expressed parametrically as* $\mathbf{r}(t) = (\cos t, \sin t, 4t)$, *where* $0 \le t \le 2\pi$.

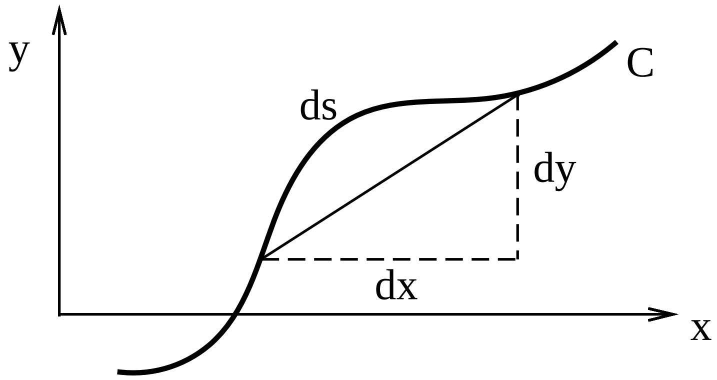

Figure 1.5: *Geometric interpretation of ds in 2D.*

1.7.1 Intrinsic Equation of a Curve

We can change parameters in a parametric formulation quite easily. Consider $\mathbf{r} = \mathbf{r}(t)$, and $t = t(u)$ with $\frac{dt}{du} > 0$ at all points, then we can describe $\mathbf{r}$ parametrically in terms of u (with the same sense as t).

For curves in space it is natural to use the arclength as parameter. From the definition of arclength $\frac{ds}{dt} \geq 0$ so $t = t(s)$ is allowable and

$$\mathbf{r} = \mathbf{r}(s) = (x(s), y(s), z(s)), \tag{1.9}$$

is an alternative parametric description of the curve called the **intrinsic equation** of the curve. If the arclength is the parameter then at any point on $\mathbf{r}(s)$ the **unit** tangent is $\frac{d\mathbf{r}}{ds}$.

[This follows from the fact that $d\mathbf{r} = (dx, dy, dz)$ so $\left|\frac{d\mathbf{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = \sqrt{\left(\frac{ds}{ds}\right)^2} = 1$.]

1.8 Curves and Surfaces

In 3D space, we can describe a curve parametrically by $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t))$, i.e. only **one** parameter is necessary in the parametric description. A surface can be represented parametrically by $\mathbf{r} = \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, i.e. **two** parameters are required to define a surface.

Examples:

- $\mathbf{r}(t) = (\cos t, \sin t, t)$ is called a circular helix (i.e. a curve), $-\infty \leq t \leq \infty$. It lies on the cylinder $x^2 + y^2 = a^2$.

- $\mathbf{r}(u, v) = (\cos u, \sin u, v)$ is a cylinder $-\infty \leq v \leq \infty$, $0 \leq u \leq 2\pi$.

1.9 Curvature

An important physical quantity when dealing with curves is their curvature. Let a curve have an **intrinsic** equation $\mathbf{r} = \mathbf{r}(s)$. The curvature of the curve at any point (i.e. at any value of s) is defined to be:

$$\kappa(s) = \left|\frac{d^2\mathbf{r}}{ds^2}\right|. \tag{1.10}$$

The quantity $\rho = \frac{1}{\kappa}$ is called the radius of curvature and corresponds to the radius of the circle that would "fit" into the curve at any point. This may of course vary from point to point.

Example 1.9.1 *Find the curvature and radius of curvature for* $\mathbf{r}(t) = (\cos t, \sin t, 4t)$, *where* $0 \leq t \leq 2\pi$.

1.10 Velocity & Acceleration

Let $\mathbf{r}(t)$ be the position vector of a point P in space, where t is time. (Just think of t as being a parameter which in this case physically corresponds to the time). Then $\mathbf{r}(t)$ represents the path C of P in space. From previous work we know that the vector function

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \tag{1.11}$$

is tangent to C and therefore points in the instantaneous direction of P. We recall also that

$$|\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right| = \frac{ds}{dt}, \tag{1.12}$$

where s is the arclength, which measures the distance of P from a fixed point ($s = 0$) on C along the curve. Hence $\frac{ds}{dt}$ is the speed of P. The vector $\mathbf{v}$ called the **velocity vector** of the motion.
The derivative of the velocity vector is called the **acceleration vector** and will be denoted $\mathbf{a}$. Thus

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}. \tag{1.13}$$

Example: Centripetal acceleration

The vector function $\mathbf{r}(t) = R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j}$ (where ω is a known constant and t is the time) represents a circle of radius R with centre at the origin in the xy-plane and describes the motion of a particle P in the anti-clockwise direction. The velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -R\omega\sin\omega t\mathbf{i} + R\omega\cos\omega t\mathbf{j}$ is tangent to C and its magnitude, the speed, is constant ($= R\omega$). The **angular speed** (speed divided by the distance R from the centre) is equal to ω. The acceleration vector is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -R\omega^2\cos\omega\mathbf{i} - R\omega^2\sin\omega\mathbf{j} = -\omega^2\mathbf{r}.$$

We see that there is an acceleration of constant magnitude $|\mathbf{a}|$ towards the origin, the so-called centripetal acceleration, which results from the fact that the velocity vector is changing direction at a constant rate. The centripetal force is thus $m\mathbf{a}$ where m is the mass of P. (Note that the opposite vector $-m\mathbf{a}$ is called the centrifugal force).

It is clear that $\mathbf{a}$ is the rate of change of $\mathbf{v}$. In the example, $|\mathbf{v}|$ is constant, but $|\mathbf{a}| \neq 0$ which illustrates that the magnitude of $\mathbf{a}$ is not in general the rate of change of $|\mathbf{v}|$. The reason is that $\mathbf{a}$ is not, in general, tangent to the path C. In fact, by applying the chain rule of differentiation to (1.11) and denoting derivatives with respect to s by primes ($'$), we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'\frac{ds}{dt},$$

and by differentiating this again

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{r}'\frac{ds}{dt}\right) = \mathbf{r}''\left(\frac{ds}{dt}\right)^2 + \mathbf{r}'\frac{d^2s}{dt^2}. \tag{1.14}$$

Since $\mathbf{r}'$ is the tangent vector and its derivative $\mathbf{r}''$ is perpendicular to $\mathbf{r}'$, the formula (1.14) is a decomposition of the acceleration vector into its normal component $\mathbf{r}''\left(\frac{ds}{dt}\right)^2$ and its tangential component $\mathbf{r}'\frac{d^2s}{dt^2}$. From this we see that if, and only if, the normal component is zero, $|\mathbf{a}|$ equals the rate of change of $|\mathbf{v}| = \frac{ds}{dt}$, except for possibly the sign, because $|\mathbf{a}| = |\mathbf{r}'|\left|\frac{d^2s}{dt^2}\right| = \left|\frac{d^2s}{dt^2}\right|$ (since $\mathbf{r}' = \frac{d\mathbf{r}}{ds}$ is the unit tangent vector).

Example: Coriolis acceleration

A particle P moves in a straight line from the centre of a disc towards the edge, the position vector being

$$\mathbf{r}(t) = t\mathbf{b}, \tag{1.15}$$

where $\mathbf{b}$ is a unit vector, rotating together with the disc with constant angular speed ω in the anti-clockwise sense. (see Figure 1.6). Find the acceleration $\mathbf{a}$ of P.

Solution. Because of the rotation, $\mathbf{b}$ is of the form

$$\mathbf{b}(t) = \cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}. \tag{1.16}$$

Differentiating (1.16) w.r.t. t we obtain the velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{b} + t\frac{d\mathbf{b}}{dt}. \tag{1.17}$$

Obviously $\mathbf{b}$ is the velocity of P relative to the disc, and $t\frac{d\mathbf{b}}{dt}$ is the additional velocity due to the rotation. Differentiating once more, we obtain the acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\frac{d\mathbf{b}}{dt} + t\frac{d^2\mathbf{b}}{dt^2}. \tag{1.18}$$

In the last term of (1.18) we have $\frac{d^2\mathbf{b}}{dt^2} = -\omega^2\mathbf{b}$ which follows from differentiating (1.16). Hence this acceleration $\frac{d^2\mathbf{b}}{dt^2}$ is directed towards the centre of the disc and from the last example we see that this is the centripetal acceleration due to the rotation. In fact, the distance of P from the centre is equal to t which therefore plays the role of R in the last example.

The most interesting and probably unexpected term in (1.18) is $2\frac{d\mathbf{b}}{dt}$, the so-called **Coriolis acceleration**, which results from the interaction of the disc and the motion of P on the disc. It has the direction of $\frac{d\mathbf{b}}{dt}$, that is, it is tangential to the edge of the disc and it points in the direction of the rotation. If P is a person of mass m walking on the disc according to (1.15), then P will feel a force $-2m\frac{d\mathbf{b}}{dt}$ in the opposite direction, that is, against the sense of rotation.

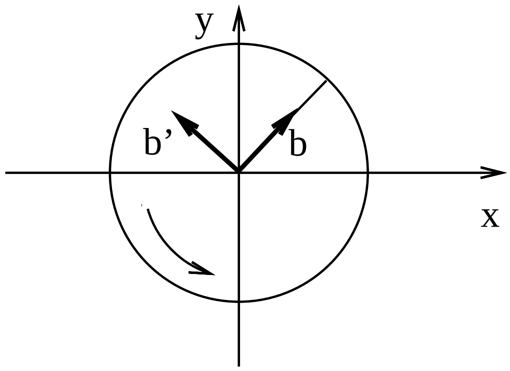

Figure 1.6: *Motion for coriolis acceleration.*

Chapter 2

Scalar & Vector Fields

2.1 Regions

Definition 2.1.1 *Let V be a set of points in space. A point $P \in V$ is an* **interior point** *of V if there exists a sphere (however small) with centre P s.t. every point of the sphere is contained in V. A point $P \in V$ is a* **boundary point** *if every sphere centred on P contains interior points and points that are not in V. The set V forms a* **region** *R if each point of V is either an interior or boundary point and if every pair of points can be joined by a continuous curve consisting entirely of points in V. R is* **open** *if it contains no boundary points. R is* **closed** *if all points not in R form one or more open regions.*

Examples:

- In 2D replace the word "sphere" in the above definition with "circle".
- $x^2 + y^2 < 1$ is an open region.
- $x^2 + y^2 \leq 1$ is a closed region; $(1, 0)$ is an example of a boundary point; $(0, 0)$ is an interior point (boundary is the circle $x^2 + y^2 = 1$).
- In 3D, $x^2 + y^2 + z^2 < 1$ is open.
- $x^2 + y^2 + z^2 \leq 1$ is a closed region; $(1, 0, 0)$ is a boundary point, $(0, 0, 0)$ is an interior point (boundary is the sphere $x^2 + y^2 + z^2 = 1$).
- Note: $1 < x^2 + y^2 \leq 2$ is neither an open region nor a closed region so a region may be neither open nor closed.

2.2 Functions of Several Variables

We are familiar with functions of one variable, e.g. $y = f(x)$, and its continuity, differentiability etc. Recall that, loosely speaking, $y = f(x)$ is continuous if it can be drawn without removing the pen from the page. This is represented by a curve in the xy-plane.
A function of two variables $z = f(x, y)$ represents a surface in 3D.
Consider a function of three variables $w = f(x, y, z)$ defined over some region of space R. $f(x, y, z)$ is continuous if no sudden jumps occur in its value as x, y, z vary in an analogous way to functions of a single variable. Most physical quantities can be represented by continuous functions, e.g. $T(x, y, z)$ may represent the temperature at each point in a room.
A function of more than two variables

$$w = f(x_1, x_2, \ldots, x_{n-1}), \tag{2.1}$$

is defined in $\mathbb{R}^n$, and is not possible to draw for $n > 3$.

2.3 Partial Derivatives

Definition 2.3.1 *The first order partial derivative* $\frac{\partial f}{\partial x}$ *of* $f(x, y, z)$ *at the point* (x, y, z) *w.r.t.* x *is defined as,*

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

with similar definitions for $\frac{\partial f}{\partial y}$ *and* $\frac{\partial f}{\partial z}$. *The partial derivatives are often written as* f_x, f_y *and* f_z.

Note: In the definition of $\frac{\partial f}{\partial x}$, the other variables y, z are held constant, so in practise we treat y, z as constants and differentiate in the usual way w.r.t. x.

Example 2.3.1 *Find the partial derivatives of* $f(x, y, z) = x^3 + x^2 y + xyz$.

Solution.

$$f_x = 3x^2 + 2xy + yz, \qquad f_y = x^2 + xz, \qquad f_z = xy.$$

To evaluate partial derivatives at a particular point, e.g. f_x at $(1, 2, 3)$ first evaluate the partial derivative symbolically first and then substitute in the values $x = 1$, $y = 2$ and $z = 3$. Thus in the above example, $f_x(1, 2, 3) = 3x^2 + 2xy + yz\big|_{(x,y,z)=(1,2,3)} = 3 + 4 + 6 = 13$.

Higher order derivatives are defined in the obvious way:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right).$$

For example, if $f = (x + 2y + 3z)^4$ then

$$f_x = 4(x + 2y + 3z)^3, \quad f_{xx} = 12(x + 2y + 3z)^2, \quad f_{xxx} = 24(x + 2y + 3z).$$

Similarly we have:

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y},$$

with a similar interpretation for f_{xz}, f_{yz} etc.

Theorem 2.3.1 *If all the mixed second order derivatives exist and are continuous at a point, then at that point:*

$$f_{xy} = f_{yx}, \qquad f_{yz} = f_{zy}, \qquad f_{zx} = f_{xz}. \tag{2.2}$$

(proof omitted).

2.3.1 Continuously differentiable functions:

The function $f(x, y, z)$ is said to be **continuously differentiable** if its first order partial derivatives f_x, f_y and f_z exist and are continuous at every point.

Example 2.3.2 *Demonstrate the above results for* $f(x, y, z) = \sin(ax + by + cz)$.

2.3.2 The Chain Rule

Suppose $F = F(f, g, h)$ is a continuously differentiable function of three variables f, g and h where $f = f(x, y, z)$, $g = g(x, y, z)$ and $h = h(x, y, z)$ are continuously differentiable functions of x, y and z. Then F is a continuously differentiable function of x, y, z and its derivatives w.r.t. x, y, z are given by

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial F}{\partial f}\frac{\partial f}{\partial x} + \frac{\partial F}{\partial g}\frac{\partial g}{\partial x} + \frac{\partial F}{\partial h}\frac{\partial h}{\partial x} \\
\frac{\partial F}{\partial y} &= \frac{\partial F}{\partial f}\frac{\partial f}{\partial y} + \frac{\partial F}{\partial g}\frac{\partial g}{\partial y} + \frac{\partial F}{\partial h}\frac{\partial h}{\partial y} \\
\frac{\partial F}{\partial z} &= \frac{\partial F}{\partial f}\frac{\partial f}{\partial z} + \frac{\partial F}{\partial g}\frac{\partial g}{\partial z} + \frac{\partial F}{\partial h}\frac{\partial h}{\partial z}.
\end{aligned} \tag{2.3}$$

Note: There is a difficulty with the notation above as we have written $F(f, g, h)$ and also in effect $F(x, y, z)$. It is strictly better to write $F(f, g, h)$ and $G(x, y, z)$ say, where $G(x, y, z) = F(f, g, h)$. This way we could write the chain rule replacing the LHS of (2.3) the chain rule with $\partial G/\partial x$, $\partial G/\partial y$ and $\partial G/\partial z$ which removes the ambiguity as $G = G(x, y, z)$.

Example 2.3.3 *Suppose that*

$$F(f, g) = f + \sin g, \qquad f = x\cos y, \qquad g = x \sin y.$$

When we write F as a function of (x, y) we should write it as a new function $G(x, y)$ with $G(x, y) = x \cos y + \sin(x \sin y)$ and $G(x, y) = F(f, g)$. If we use this notation, no ambiguity arises when we write partial derivatives. Directly from this definition we therefore have

$$\frac{\partial G}{\partial x} = \cos y + \cos(x \sin y) \sin y. \tag{2.4}$$

Now, from the chain rule

$$\frac{\partial G}{\partial x} = \frac{\partial F}{\partial f}\frac{\partial f}{\partial x} + \frac{\partial F}{\partial g}\frac{\partial g}{\partial x} = \cos y + \cos g \sin y = \cos y + \cos(x \sin y) \sin y,$$

and this equals (2.4).

2.4 Definitions of Scalar & Vector Fields

Definition 2.4.1 *Suppose a scalar $\Omega(x, y, z)$ is defined on a point set U in 3D space, i.e. to each point $P(x, y, z)$ in U there corresponds a single scalar value of Ω. Then Ω is called a scalar function of position or a* **scalar field**. *Likewise if a vector $\mathbf{v}(x, y, z)$ is defined on a point set U, then $\mathbf{v}$ is called a vector function of position or a* **vector field**. *(U will usually be a region). Alternative notation for $\Omega(x, y, z)$ and $\mathbf{v}(x, y, z)$ is $\Omega(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ where $\mathbf{r}$ is the position vector for the point $P(x, y, z)$. Note that $\mathbf{v}$ has three components like any vector.*

We emphasise the notation for the **position vector** which we will give the special label $\mathbf{r}$. We will often write $\mathbf{r} = (x, y, z)$ and by this we strictly mean the vector with components $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, i.e. the vector joining the point $(0, 0, 0)$ to the general point with coordinates (x, y, z). On occasion however we will also use $\mathbf{r}$ to refer to the point with coordinates (x, y, z).

Example 2.4.1 *In a flowing liquid, the velocity field might be given by $\mathbf{u} = (z, x + y, x + zy)$ and the pressure by $p = x + z$. Thus $\mathbf{u}$ is a vector field and p is a scalar field. What is $\mathbf{u}$ at the point with position vector $\mathbf{r} = (0, 0, 1)$? What direction is the flow at this point? What is the pressure at $\mathbf{r} = (1, 1, 1)$?*

2.5 Gradient of a Scalar Field

Definition 2.5.1 *If $f(x, y, z)$ is defined and continuously differentiable in some open region R then the* **gradient** *of f is defined as:*

$$grad\, f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \tag{2.5}$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Example 2.5.1 *Find ∇f when $f(x, y) = x^2 + xy + y^2$.*

2.6 Properties of the Gradient (of a scalar)

2.6.1 The directional derivative

We are used to finding (partial) derivatives w.r.t. the co-ordinate axes x, y, z. Consider a scalar function of a single variable, e.g. $y = f(x)$. Geometrically this is represented by a curve and the only possible derivative is $\frac{dy}{dx}$, and it denotes the rate of change of y in the direction of increasing x. Suppose we now have a scalar function of two variables, e.g. $z = f(x, y)$. In this case we have two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Geometrically these correspond to the rate of change of $f(x, y)$ in the directions x and y respectively.

Consider the example in Figure 2.1, showing a function of two variables, and fix attention on some point on the surface represented by $z = f(x, y)$. At this point the partial derivative reflects the rate of change in the directions x and y of the co-ordinate axes. If a cyclist were located at the point in question, then if they faced in the positive x direction, the slope of the land immediately ahead would be given by $\frac{\partial f}{\partial x}$ while if they faced in the positive y direction the slope would be given by $\frac{\partial f}{\partial y}$. When we move to functions of three independent variables, it is not so easy to build up a geometric picture, but the picture for functions of two variables is sufficient to understand what follows. We would like to generalise our partial derivatives so that we can obtain expressions for the rate of change of the scalar function under consideration *in any direction.* In Figure 2.1 we wish to allow the cyclist to face in **any** direction (not just parallel to the co-ordinate axes) and still be able to estimate the slope in that particular direction.

Let $f(x, y, z)$ be a scalar field. We define the directional derivative of f in the direction of *any* vector $\mathbf{n}$ in the following way. Let P be a fixed point and P' another point which varies in such a way that the vector $\overrightarrow{PP'}$ is always parallel to a fixed unit vector $\hat{\mathbf{n}}$. Assume that the scalar field f takes the values $f(P)$, $f(P')$ at P and P' respectively. The derivative of f at P in the direction of $\hat{\mathbf{n}}$, which we denote $\frac{\partial f}{\partial n}$, is defined as

$$\frac{\partial f}{\partial n} := \lim_{P \to P'} \frac{f(P') - f(P)}{PP'},$$

wherever the limit exists. It is clear that, in general, f will vary at different rates as we move away from P in different directions; the directional derivative measures the rate of variation in the direction of $\hat{\mathbf{n}}$.

The most **important formula** involving the directional derivative is:

$$\frac{\partial f}{\partial n} = \text{grad}\, f \cdot \hat{\mathbf{n}} = \nabla f \cdot \hat{\mathbf{n}}, \tag{2.6}$$

i.e. it is the component of grad f in the direction of $\hat{\mathbf{n}}$.

The notation $\frac{\partial f}{\partial n}$ might be misleading: we are not differentiating f w.r.t. n!

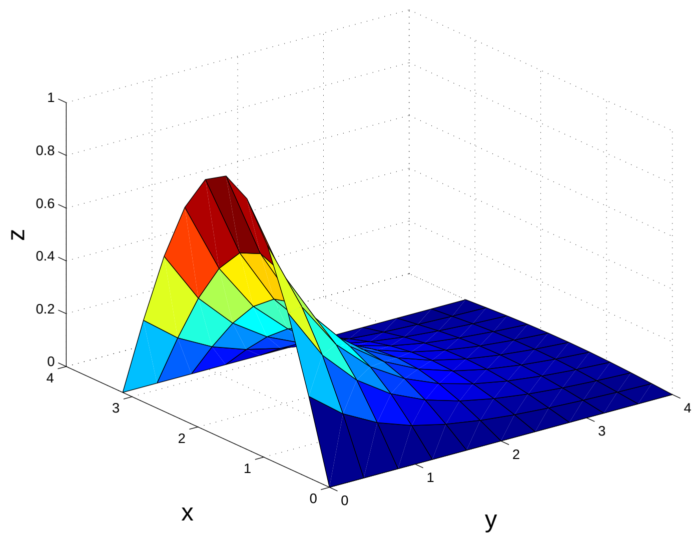

Figure 2.1: *Graph of the surface* $z = f(x, y) = \sin x e^{-y}$ *for* $0 \leq x \leq \pi$, $0 \leq y \leq 4$.

Example 2.6.1 *Find the directional derivative of* $f(x, y, z) = 2x^2 + 3y^2 + z^2$ *at the point* $P(2, 1, 3)$ *in the direction of the vector* $\mathbf{n} = \mathbf{i} - 2\mathbf{k}$.

Example 2.6.2 *Find the directional derivative of* $f(x, y, z) = x^2y^2z^2 + x + y + z$ *in the direction of the vector* $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ *at the point* $(1, 1, 1)$.

2.6.2 Direction of maximum increase

Now

$$\frac{\partial f}{\partial n} = \nabla f \cdot \hat{\mathbf{n}} = |\nabla f||\hat{\mathbf{n}}| \cos \theta,$$

where θ is the angle between the vectors ∇f and $\hat{\mathbf{n}}$. Clearly the directional derivative has its largest value when $\theta = 0$ (i.e. when $\cos \theta = 1$). This is the **direction of maximum increase** and now

$$\frac{\partial f}{\partial n} = |\nabla f|.$$

Example 2.6.3 *The flow of heat in a temperature field takes place in the direction of maximum decrease of temperature* $T = x/y$. *Find this direction at the point* $P(8, -1)$.

2.6.3 Level curves and surfaces

Consider a scalar field $z = f(x, y)$, this represents a surface in 3D. At points where f has a maxima/minima we will have mountains/valleys. We can also represent $f(x, y)$ in a 2D representation by drawing *contours* or *level curves* which are curves along which f is a constant. For example, if f has a local maximum (mountain top) the contours or level curves might look like those in Figure

2.2. We draw contours by setting $f(x, y) = c$, a parameter, and finding all the points (x, y) which are at this particular height. By varying c we can draw all the contours and get and idea of what the scalar field looks like without having to draw a 3D picture.
Another example is a weather map on which isobars are lines of equal pressure p, which is a scalar function $p = p(x, y)$ where (x, y) represent the location on the earth's surface. We can picture what the surface representing the pressure $p(x, y)$ would look like from a consideration of the contours: a centre of high pressure would be geometrically similar to a mountain top.

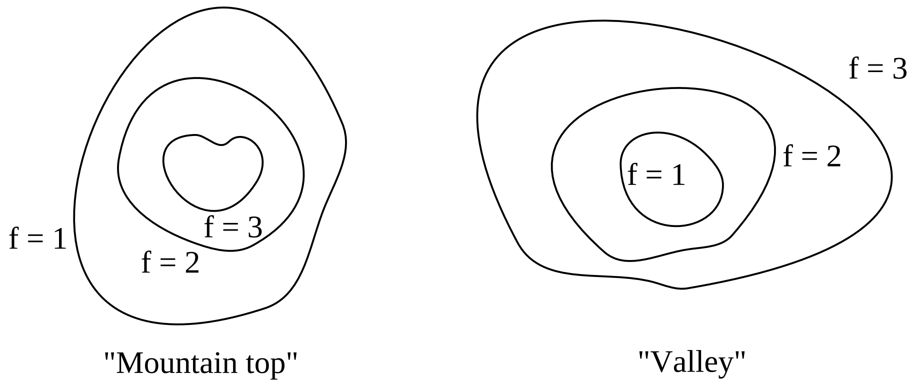

Figure 2.2: *Contours for a mountain-top and a valley.*

Example 2.6.4 *Draw the contours $f(x, y) = x^2 + y^2$.*

In 3D with $w = f(x, y, z)$, we cannot draw this in 4D. However, if we generalise the above, then the *contours now become level surfaces* defined by $f(x, y, z) = c$. As c varies we get different contours each of which is now a *surface* in space.

Example 2.6.5 *What are the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$?*

2.6.4 Important connection between f and its level surface

Theorem 2.6.1 *Suppose $f(x, y, z)$ is a scalar field. Consider a level surface given by $f(x, y, z) = c$ and choose a particular point $P(x, y, z)$ on the level surface. If grad f does not vanish at this point, then the vector grad f is* **normal** *to the level surface $f = c$ at P.*

Proof. Consider a curve in space passing through P and lying on the level surface $f = c$. This curve (call it C) may be written *parametrically* as $\mathbf{r}(t) = (x(t), y(t), z(t))$. As C lies on the surface $f = c$ we have $f(x(t), y(t), z(t)) = c$. Differentiating this equation by the chain rule gives:

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{dc}{dt} = 0.$$

That is,

$$\nabla f \cdot \frac{d\mathbf{r}}{dt} = 0.$$

Now $\mathbf{r}$ is a curve on the surface $f = c$ and $\frac{d\mathbf{r}}{dt}$ is tangent to the curve defined by $\mathbf{r}$. From the results in Chapter 1 we know that it is perpendicular to the vector $\mathbf{r}(t)$. But $\mathbf{r}(t)$ is any curve on the surface $f = c$ and so grad $f = \nabla f$ must be perpendicular to the tangent plane of the level surface $f = c$, i.e. ∇f is orthogonal to all vectors $\frac{d\mathbf{r}}{dt}$ in the tangent plane. Thus ∇f is perpendicular to its own level surfaces.

This is a very useful result as we can apply it to **any** surface or curve. Suppose we have a curve given by $y = f(x)$ or a surface given by $z = g(x, y)$. Then we rewrite each formula as $y - f(x) = 0$ and $z - g(x, y) = 0$ and the theorem tells us that the normal vector to the curve or surface is $\nabla(y - f(x))$ and $\nabla(z - g(x, y))$ respectively.

Example 2.6.6 *Consider a surface given by* $f(x, y) = \ln(x^2 + y^2)$. *Demonstrate the above result.*

Solution. We need to show that ∇f is normal to the level curves of f. The level surfaces (curves in this case) are given by

$$f = c, \quad \text{c arbitrary} \qquad \Longrightarrow \qquad \ln(x^2 + y^2) = c \qquad \Longrightarrow \qquad x^2 + y^2 = e^c = \mu,$$

where $\mu = e^c$ is arbitrary. So the level surfaces are defined by $x^2 + y^2 = \mu$, for arbitrary μ. Thus the level curves are circles.

Consider the case $\mu = 4$ and examine the point $(2, 0)$ on the level curve $x^2 + y^2 = 4$. At $(2, 0)$, the normal to this curve is $\mathbf{i}$ (i.e. the unit vector in the x direction). In addition

$$\nabla f = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right),$$

so at the point $(2, 0)$, $\nabla f = (1, 0) = \mathbf{i}$ and so $\nabla f|_{(1,0)}$ is in the direction of the normal to the curve at this point. This is represented geometrically in Figure 2.3.

Example 2.6.7 *Find a unit normal vector for the curve* $y = 1 - x^2$ *at* $P = (1, 0)$.

2.6.5 Taylor's Expansion

For a function of one real variable $y = f(x)$, Taylor's expansion allows us to write down the value of a function near a point x_0, solely in terms of its value of x_0 and derivatives of the function at x_0, i.e. solely in terms of quantities evaluated at the point x_0.

Thus if $f(x)$ is a well behaved function:

$$f(x) = f(x_0) + h \left. \frac{df}{dx} \right|_{x=x_0} + \frac{h^2}{2} \left. \frac{d^2 f}{dx^2} \right|_{x=x_0} + \mathcal{O}(h^3), \tag{2.7}$$

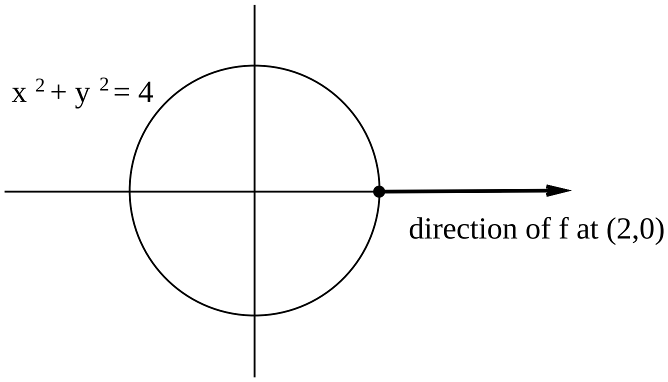

Figure 2.3: *2D example of the relationship between the level surfaces of* f *and* ∇f.

where $h = x - x_0$. Such expansions are very useful if $h \ll 1$ as we can usually truncate after two terms. [Recall that if $h \ll 1$, then h^2 and higher powers are even smaller so that for $h \ll 1$ the first two (or three) terms on the RHS will be a good approximation for the LHS].

For a function of several variables $f(x, y, z)$, the analogous result is:

$$f(x,y,z) = f(x_0,y_0,z_0) + h\,\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0,z_0)} + k\,\frac{\partial f}{\partial y}\bigg|_{(x_0,y_0,z_0)} + l\,\frac{\partial f}{\partial z}\bigg|_{(x_0,y_0,z_0)} + \mathcal{O}(h^2+k^2+l^2), \quad (2.8)$$

where $h = x - x_0$, $k = y - y_0$ and $l = z - z_0$.

So, we have an expression for f at some point removed from (x_0, y_0, z_0) in terms of quantities evaluated solely at (x_0, y_0, z_0). This approximation is most useful for $h, k, l \ll 1$.

Note if we define the vector $(h, k, l) = \delta\mathbf{r}$, then we can write the above expression in vector notation:

$$f(x,y,z) = f(x_0,y_0,z_0) + \delta\mathbf{r}\cdot\nabla f|_{(x_0,y_0,z_0)} + \mathcal{O}(|\delta\mathbf{r}|^2).$$

Example 2.6.8 *Use Taylor's expansion to find a first order approximation for* $f(1.5, 2.5)$ *based on quantities estimated only at the point* $(1, 3)$ *if* $f(x, y) = x^2 + y^2$. *What is the error in your estimate?*

2.7 Divergence and Curl of a Vector Field

Suppose $\mathbf{f}$ is a continuously differentiable vector field

$$\mathbf{f} = \mathbf{f}(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)),$$

in some open region R.

Then, in R, the **divergence of the vector field** $\mathbf{f}(x, y, z)$ is defined to be the **scalar** quantity:

$$\operatorname{div}\mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}. \quad (2.9)$$

In index notation this can be represented as $\frac{\partial f_i}{\partial x_i}$.

The **curl of the vector field** $\mathbf{f}(x, y, z)$ is defined to be the vector with components:

$$\operatorname{curl} \mathbf{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right), \tag{2.10}$$

which can be more easily expressed in the following form:

$$\operatorname{curl} \mathbf{f} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{pmatrix}, \tag{2.11}$$

and so $\operatorname{curl} \mathbf{f}$ is a vector field.

Note:

1. In fluids every flow of an incompressible fluid must satisfy $\operatorname{div} \mathbf{v} = 0$ where $\mathbf{v}(x, y, z)$ is the velocity at any point in the fluid.

2. A vector field for which $\operatorname{div} \mathbf{v} = 0$ is said to be a **divergence-free vector field**.

3. If $\mathbf{v}(x, y, z)$ is the velocity of a fluid, then $\operatorname{curl} \mathbf{v}$ is termed the **vorticity** and concerns whether or not fluid particles rotate.

4. If $\operatorname{curl} \mathbf{v} = 0$ **everywhere** the flow is termed irrotational and fluid particles do not rotate.

Example 2.7.1 *Find the divergence of* $\mathbf{f}$ *if (i)* $\mathbf{f} = (x^2 + 2y + z)\mathbf{i} + (3y)\mathbf{j} + (x^3 + y)\mathbf{k}$, *(ii)* $\mathbf{f} = \mathbf{r} = (x, y, z)$.

Example 2.7.2 *Find curl* $\mathbf{f}$ *and curl curl* $\mathbf{f}$ *where* $\mathbf{f} = (z + x, x + y, y + z)$.

Note: The above example shows that for a *constant* vector field $\mathbf{F}$, we always have $\operatorname{curl} \mathbf{F} = \mathbf{0}$ (which can easily be seen from (2.11) since the partial derivatives are zero if f_1, f_2, f_3 are constant).

2.7.1 Physical Interpretation of the Divergence

Imagine a compressible fluid in a two dimensional flow with density $\rho = \rho(x, y)$ and velocity vector field $\mathbf{u}(x, y) = (u(x, y), v(x, y))$, i.e. in this simple case the velocity vector only has two components. Consider a small element in the flow domain of dimensions Δx and Δy and imagine an amount of liquid entering and leaving the element in unit time (i.e. per second), see Figure 2.4.

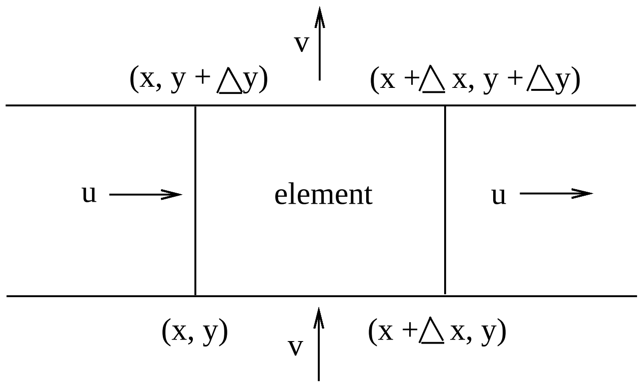

Figure 2.4: *Small element in the flow domain.*

The mass of liquid passing through the left hand upright edge in the positive x direction in unit time (mass flux) is approximately

$$\rho(x, y + \Delta y/2)u(x, y + \Delta y/2)\Delta y.$$

The mass of liquid passing through the right hand upright edge in the positive x direction in unit time is

$$\rho(x + \Delta x, y + \Delta y/2)u(x + \Delta x, y + \Delta y/2)\Delta y.$$

Thus the net mass flow of liquid passing **out** (through both the left hand and right hand edges) of the element per unit time is

$$\begin{aligned}&\big[\rho(x + \Delta x, y + \Delta y/2)u(x + \Delta x, y + \Delta y/2) - \rho(x, y + \Delta y/2)u(x, y + \Delta y/2)\big]\Delta y\\ &= \frac{\big[\rho(x + \Delta x, y + \Delta y/2)u(x + \Delta x, y + \Delta y/2) - \rho(x, y + \Delta y/2)u(x, y + \Delta y/2)\big]}{\Delta x}\Delta x\Delta y.\end{aligned}$$

From elementary calculus if $\rho(x, y)u(x, y)$ is a function of two variables x, y then:

$$\frac{\partial(\rho u)}{\partial x} = \lim_{\Delta x \to 0} \frac{\big[\rho(x + \Delta x, y + \Delta y/2)u(x + \Delta x, y + \Delta y/2) - \rho(x, y + \Delta y/2)u(x, y + \Delta y/2)\big]}{\Delta x},$$

and so letting the size of the fluid element tend to zero (i.e. letting $\Delta x \to 0$ and $\Delta y \to 0$) we arrive at the result:

net mass flow rate of liquid out of element through vertical edges as $\Delta x \to 0$ is $\frac{\partial(\rho u)}{\partial x}\Delta x\Delta y$.

By a similar argument:

net mass flow through the horizontal edges is $\frac{\partial(\rho v)}{\partial y}\Delta x\Delta y$.

Hence the net flow of liquid out of the (infinitesimally) small element of area $dx\,dy$ is given by

$$\frac{\partial(\rho u)}{\partial x}dx\,dy + \frac{\partial(\rho v)}{\partial y}dx\,dy = \text{div}(\rho\mathbf{u})dx\,dy = \text{div}(\rho\mathbf{u})dV,$$

where dV is the "volume" of the element (in this 2D problem it is the area of the element). This is the origin of the term **divergence**: it refers to the amount of some quantity **diverging** out of each point in the region under consideration.

The above argument can be generalised to 3D problems where the flow occurs in three dimensions. The following equivalent expression results (with $\mathbf{u}$ now a vector with 3 components):

The net flow of liquid out of an element $dx\,dy\,dz$ per unit time equals

$$\operatorname{div}(\rho(x,y,z)\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{z}))\,dx\,dy\,dz.$$

If the liquid is incompressible, then $\rho(x,y,z)$ is constant and so $\operatorname{div}(\rho\mathbf{u}) = \rho\operatorname{div}\mathbf{u}$. But by **conservation of mass**, the same **mass** of liquid must be flowing into the element as out of it (as there is no change in density in the liquid element) and so for an incompressible liquid $\rho\operatorname{div}\mathbf{u} = 0$ and hence $\operatorname{div}\mathbf{u} = 0$. If $\mathbf{u}$ represents the velocity vector field for any **incompressible** flow, then necessarily $\operatorname{div}\mathbf{u} = 0$ **everywhere in the flow**.

2.7.2 Physical interpretation of curl

A similar sort of analysis can be performed for a flowing liquid to show that if $\mathbf{u}$ is the velocity vector, then $\operatorname{curl}\mathbf{u}$ at any point is a measure of the tendency of the fluid element at that point to rotate. In principle $\operatorname{curl}\mathbf{u}$ could be measured by inserting a little paddle wheel into the fluid at any point. The rotation of the wheel would be a measure of the curl.

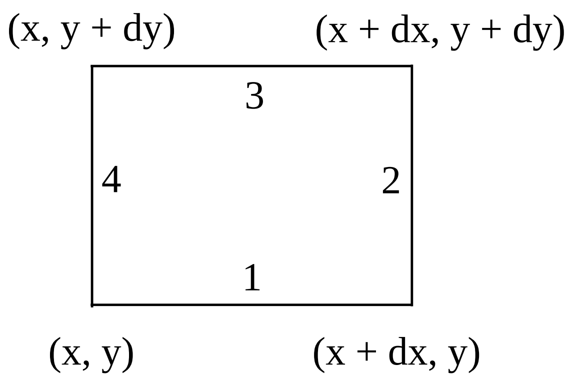

Figure 2.5: *Small element in the flow domain.*

To see this imagine a liquid flowing (in 2D) with velocity vector $\mathbf{u} = (u, v)$. Consider now a small square in the flowing liquid as in Figure 2.5. The circulation of the velocity vector about the square indicates the tendency of the liquid to move around the square and is defined by:

$$\begin{aligned}\int_{\text{square}} \mathbf{u}\cdot\mathbf{dr} &= \int_{\text{side 1}} u\,dx + \int_{\text{side 2}} v\,dy - \int_{\text{side 3}} u\,dx - \int_{\text{side 4}} v\,dy \\ &= \text{calculation of integral about the square 1234.}\end{aligned}$$

Now write all the velocity components as Taylor series about (x, y) while assuming velocity along 1 is (approximately) $\mathbf{u}(x + dx/2, y)$, along 2 is $\mathbf{u}(x + dx, y + dy/2)$, along 3 is $\mathbf{u}(x + dx/2, y + dy)$ and along 4 is $\mathbf{u}(x, y + dy/2)$. Thus, using Taylor series to first order:

$$\begin{aligned}
&\text{along 1:} \quad \mathbf{u}(x + dx/2, y) = u(x + dx/2, y) \approx u(x, y) + \frac{dx}{2}\frac{\partial u}{\partial x}\\
&\text{along 2:} \quad \mathbf{u}(x + dx, y + dy/2) = v(x + dx, y + dy/2) \approx v(x, y) + dx\frac{\partial v}{\partial x} + \frac{dy}{2}\frac{\partial v}{\partial y}\\
&\text{along 3:} \quad \mathbf{u}(x + dx/2, y + dy) = u(x + dx/2, y + dy) \approx u(x, y) + \frac{dx}{2}\frac{\partial u}{\partial x} + dy\frac{\partial u}{\partial y}\\
&\text{along 4:} \quad \mathbf{u}(x, y + dy/2) = v(x, y + dy/2) \approx v(x, y) + \frac{dy}{2}\frac{\partial v}{\partial y}.
\end{aligned}$$

Note that the velocity vector is $\mathbf{u}(x, y) = (u(x, y), v(x, y))$. In the direction of side 1 the component of the velocity vector is just $u(x, y)$. Similarly in the direction of side 2 the component is just $v(x, y)$ etc. Thus

$$\begin{aligned}\int_{\text{square}} \mathbf{u}\cdot\mathbf{dr} &\approx \left(u + \frac{dx}{2}\frac{\partial u}{\partial x}\right)dx + \left(v + dx\frac{\partial v}{\partial x} + \frac{dy}{2}\frac{\partial v}{\partial y}\right)dy \\ &\quad - \left(u + \frac{dx}{2}\frac{\partial u}{\partial x} + dy\frac{\partial u}{\partial y}\right)dx - \left(v + \frac{dy}{2}\frac{\partial v}{\partial y}\right)dy \\ &\approx \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy.\end{aligned}$$

But $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the third component of $\operatorname{curl}\mathbf{u}$, see (2.10), and so the circulation per unit area (as $dx\,dy$ is the area of the square) is the third component of $\operatorname{curl}\mathbf{u}$. We can also interpret this as meaning that the circulation about an infinitesimal area equals the component of the curl normal to the area (as the square in Figure 2.5 was in the (x, y) plane while the circulation per unit area was the component of the curl in the z-direction).

Vector fields for which the curl is zero **everywhere** are said to be curl-free or **irrotational** vector fields.

Example 2.7.3 *A fluid has velocity field* $\mathbf{u} = (x+z, y^2, 0)$. *Check whether the flow is incompressible and/or irrotational.*

2.8 The Del Operator

An equation like $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$ is sometimes written as

$$\left(\frac{d^2}{dx^2} + 2\frac{d}{dx} + 3\right)y = 0 \quad \text{or} \quad (D^2 + 2D + 3)y = 0 \quad \text{with } D \equiv \frac{d}{dx}.$$

D is called an operator and it needs an operand to make sense, i.e it cannot stand alone. So, for example

$$D(y) = \frac{dy}{dx}.$$

.

Note that the D operator obeys some but not all the rules of ordinary algebra, e.g. $D(uv) \neq uD(v) \neq uvD$. In fact, $D(uv) = uD(v) + vD(u)$ which is just the product rule for differentiation.

2.8.1 The Del Operator in Cartesian Coordinates

The expression

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \tag{2.12}$$

is called the **del operator** or **del** or **nabla**.

Under rotation of axes (or translation) ∇ behaves like a vector and is termed a **vector operator** (compare D above). Thus we formally define:

$$\nabla\Omega = \text{grad } \Omega, \tag{2.13}$$

and we think of ∇ as operating on the scalar field $\Omega(x, y, z)$. Similarly we define (for the vector field $\mathbf{v}(x, y, z)$):

$$\nabla \cdot \mathbf{v} = \text{div } \mathbf{v}, \tag{2.14}$$

and likewise

$$\nabla \times \mathbf{v} = \text{curl } \mathbf{v}. \tag{2.15}$$

As with the D operator, **the components of ∇ act only upon the function to their right**.

Chapter 3

Line, Surface & Volume Integrals

Typically a **definite** integral will look like $\int_a^b f(x)\,dx$ where $f(x)$ is called the *integrand* and a, b are the limits of integration which physically corresponds to getting the area under the curve between a and b (see Figure 3.1). This integral is defined as follows:

$$\int_a^b f(x)\,dx = \lim_{m\to\infty} \sum_{i=1}^{m} f(x_i)\delta x_i. \tag{3.1}$$

Integration is merely a **limiting summation** and for more advanced types of integration, we define everything in an analogous fashion.

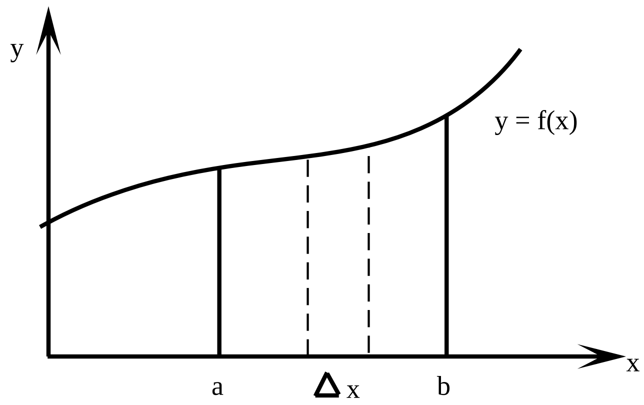

Figure 3.1: $\int_a^b f(x)\,dx =$ *area under curve* $y = f(x)$ *between* $x = a$ *and* $x = b$.

In practice, one often uses the fact that integration and differentiation are inverse operations to carry out integrations, e.g. $\int \sin x\,dx = -\cos x$ because $\frac{d}{dx}(\cos x) = -\sin x$. It is important to appreciate that this is a useful way of evaluating many elementary integrals but that integration is actually defined as the limiting summation in (3.1). If one knows the value of the function $f(x)$ at all values of $x \in [a, b]$, then in principle one can evaluate the area under the curve in Figure 3.1 (i.e. estimate the value of the definite integral) whether or not one knows the antiderivative of the integrand.

As the definite integral is the most basic type of integral the usual strategy in evaluating more complicated integrals (to be introduced in this chapter) is by **reduction to definite integrals** by some means or another.

3.1 Line Integral of a scalar field

This is a generalisation of the definite integral. Consider a piecewise smooth curve in space C with **intrinsic** parametric equation

$$\mathbf{r} = \mathbf{r}(s) = (x(s), y(s), z(s)), \qquad 0 \leq s \leq l, \tag{3.2}$$

where s is the arclength along the curve. Suppose that some scalar function $\Omega(x, y, z)$ is defined at every point of C. We break C up into little strips (see Figure 3.2) and consider Ω as being approximately constant over each strip.

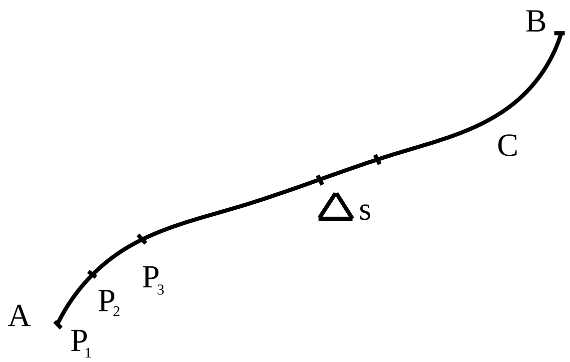

Figure 3.2: *Breaking the curve C into strips.*

Now consider the sum:

$$\Omega(P_1)\Delta s + \Omega(P_2)\Delta s + \ldots + \Omega(P_n)\Delta s = \sum_{i=1}^{n} \Omega(P_i)\Delta s$$

and define the limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Omega(P_i)\Delta s = \int_C \Omega(x, y, z)\, ds = \int_C \Omega(s)\, ds, \tag{3.3}$$

to be the line integral of the scalar function $\Omega(x, y, z)$ along the curve C, called the path of integration. The last equality comes from the fact that because the curve C is represented by (3.2), then along the curve each of x, y, z is a function of s and so $\Omega = \Omega(s)$ along C also. In (3.3) it is possible to put limits on the integration corresponding to the values of s at the start and end points of the curve C. Thus the integral can also be written as:

$$\int_{s_0}^{s_1} \Omega(s) ds, \tag{3.4}$$

where s_0, s_1 are the values of the arclength corresponding to the two ends of the curve (typically $s_0 = 0$ if we choose to measure the distance from this point).

The normal way of evaluating a line integral is by obtaining a **parametric representation** for the path of integration C, that is C: $x = x(t)$, $y = y(t)$, $z = z(t)$, with t varying, i.e. $\alpha \leq t \leq \beta$. Then

$$\int_C \Omega(x,y,z)ds = \int_\alpha^\beta \Omega(x(t),y(t),z(t))\frac{ds}{dt}dt$$
$$= \int_\alpha^\beta \Omega(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt, \tag{3.5}$$

using the expression in (1.7) to replace $\frac{ds}{dt}$. The method is best illustrated by an example.

Example 3.1.1 *Evaluate* $\int_C \Omega\, ds$ *where* $\Omega(x,y,z) = xy^3$ *and* C *is the segment of the line* $y = 2x$, $z = 0$ *in the* xy *plane from* $(-1,-2,0)$ *to* $(1,2,0)$.

Summarising: line integrals are evaluated by obtaining a parametric representation for the path of integration $\mathbf{r} = \mathbf{r}(t)$, using the fact that $ds = \frac{ds}{dt}dt$ and then writing the integrand Ω and $\frac{ds}{dt}$ in terms of t reducing it to a definite integral.

Example 3.1.2 *Evaluate* $\oint_C \Omega\, ds$ *when* $\Omega = x^2 + y^2$ *and* C *is the triangle with vertices at* $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ *(see Figure 3.3).*

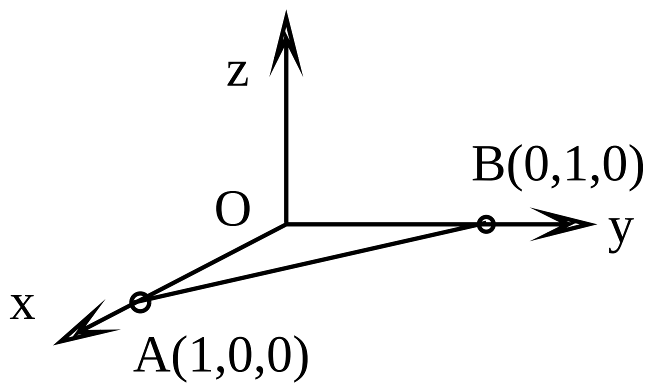

Figure 3.3: *Triangle with vertices at* $(0,0,0)$, $(1,0,0)$, $(0,1,0)$.

Remarks:

(i) The line integral along a curve C is independent of the direction along which the curve is traversed **as long as arclength is taken to be increasing when we move from the designated startpoint to the designated endpoint**, i.e.

$$\int_C \Omega\, ds = \int_{-C} \Omega\, ds.$$

(ii) For a line integral about a **closed** curve (for example a circle) the value of the integral is independent of the point at which one starts. Such integrals are often written as $\oint_C \Omega\, ds$ to emphasize the fact that the integration curve is closed.

(iii) Analogous to definite integrals we have $\int_C k\Omega\, ds = k\int_C \Omega\, ds$ where k is a constant. In addition (as occurred in the previous example) it is possible to split up the range of integration and add the constituent parts together:

$$\int_C \Omega\, ds = \int_{C_1} \Omega\, ds + \int_{C_2} \Omega\, ds,$$

where $C = C_1 + C_2$. Finally, analogous to definite integrals we have:

$$\int_C (f+g)\, ds = \int_C f\, ds + \int_C g\, ds.$$

Exercise: Evaluate the line integral of $\Omega = (a^2y^2/b^2 + b^2x^2/a^2)^{1/2}$ around the ellipse $x^2/a^2 + y^2/b^2 = 1$, $z = 0$, where a, b are known constants. [Hint: The parametric equations of the ellipse are: $x = a\cos\theta$, $y = b\sin\theta$, $z = 0$, for $0 \le \theta \le 2\pi$].

3.2 Line Integrals of a Vector Field

Let $\mathbf{f}$ be a vector defined at all points on a piecewise smooth curve C. Let $\hat{\mathbf{t}}$ denote the unit tangent along C. We define:

$$I = \int_C \mathbf{f}\cdot\hat{\mathbf{t}}\, ds, \tag{3.6}$$

to be the scalar line integral of $\mathbf{f}$ along C. (It is a scalar because of the dot product).
If $\mathbf{r} = \mathbf{r}(s)$ is the position vector of any point on C then

$$\hat{t} = \frac{d\mathbf{r}}{ds}, \tag{3.7}$$

and we can write the integral as:

$$I = \int_C \mathbf{f}\cdot\frac{d\mathbf{r}}{ds}\, ds = \int_C \mathbf{f}\cdot d\mathbf{r}, \tag{3.8}$$

(where $\mathbf{r} = (x, y, z)$ is the usual position vector and $d\mathbf{r} = (dx, dy, dz)$).
If the curve C is closed the integral is called the **circulation of the vector f** about the curve and is written:

$$\text{Circulation} = \oint_C \mathbf{f}\cdot d\mathbf{r}. \tag{3.9}$$

In fluid mechanics if $\mathbf{u}$ is the velocity field vector the circulation about a closed curve i.e. $\oint_C \mathbf{u}\cdot d\mathbf{r}$ indicates the tendency of fluid elements to move around the curve C.
Note that the direction in which the integration is carried out in a line integral **of a vector field** is important. If we reverse the direction of integration $\hat{\mathbf{t}}$ is also reversed and the integral changes sign.

So, for example if we are talking about the scalar line integral of a vector field about a circle, we must define the direction in which the integration is to be carried out (clockwise or anticlockwise). The technique for evaluating scalar line integrals of a vector field is similar to that used for line integrals of a scalar field. A parametric definition of the curve C must be found in the form $\mathbf{r} = \mathbf{r}(t)$ (note that this parameter t is a *dummy parameter*: we could have just as easily written $\mathbf{r} = \mathbf{r}(\theta)$). Then the integrand is written completely in terms of the parameter t. Finally, we use the fact that $\mathbf{dr} = \frac{d\mathbf{r}}{dt}dt = \mathbf{r}'(t)dt$ (compare with the equivalent step for writing ds in terms of dt in the case of the line integral of a scalar field).

Example 3.2.1 *Evaluate* $\oint_C \mathbf{f} \cdot \mathbf{dr}$ *for the vector field* $\mathbf{f} = (z, x, y)$ *along the curve* C = *the circle* $x^2 + y^2 = a^2$, $z = 0$, *described in a clockwise sense for an observer looking along the positive* z*-axis.*

3.2.1 Work

In mechanics, if a force $\mathbf{f}$ moves its point of application along a curve C in doing work then the amount of **work done** is given by the line integral:

$$\text{Work done} = \int_C \mathbf{f} \cdot \mathbf{dr}.$$

Example 3.2.2 *Find the work done in moving a particle in a force field given by*

$$\mathbf{F}(x, y, z) = 3xy\mathbf{i} - 5z\mathbf{j} + 10x\mathbf{k},$$

along the curve with parametric definition $\mathbf{r}(t) = (t^2 + 1, 2t^2, t^3)$, *for* $1 \le t \le 2$.

3.2.2 Conservative Fields

From introductory calculus we had the **Fundamental Theorem of Calculus** which told us how to evaluate definite integrals, i.e.

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

It turns out that there is a version of this for line integrals over certain kinds of vector fields:

Theorem 3.2.1 *Suppose that* C *is a smooth curve given by* $\mathbf{r}(t)$, *for* $a \le t \le b$. *Also suppose that* f *is a function whose gradient vector,* ∇f, *is continuous on* C. *Then*

$$\int_C \nabla f \cdot \mathbf{dr} = f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big). \tag{3.10}$$

[We omit the proof].

Example 3.2.3 *Evaluate* $\int_C \nabla f \cdot \mathbf{dr}$ *where* $f(x, y, z) = \cos \pi x + \sin \pi y - xyz$ *and* C *is any path that starts at the point* $(1, \frac{1}{2}, 2)$ *and ends at* $(2, 1, -1)$.

Solution. First notice that we did not specify the path for getting from the initial point to the end point. The reason for this is simple: Theorem 3.2.1 tells us that all we need are the initial and end points on the curve in order to evaluate this kind of line integral. Thus

$$\begin{aligned}\int_C \nabla f \cdot \mathbf{dr} &= f(2, 1, -1) - f\left(1, \frac{1}{2}, 2\right) \\ &= \cos 2\pi + \sin \pi - (2)(1)(-1) - \left[\cos \pi + \sin \frac{\pi}{2} - (1)\left(\frac{1}{2}\right)(2)\right] = 4.\end{aligned}$$

The important idea from this example is that, for these kinds of line integrals, we did not need to know the path to get the answer. In other words, we could use any path we want and we will always get the same results.

Definition 3.2.1 *(i)* $\mathbf{F}$ *is a* **conservative vector field** *if there is a scalar function* ϕ *such that* $\mathbf{F} = \nabla \phi$. *The function* ϕ *is called a* **potential function** *for the vector field.*

(ii) $\int_C \mathbf{F} \cdot \mathbf{dr}$ *is* **independent of path** *if* $\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = \int_{C_2} \mathbf{F} \cdot \mathbf{dr}$ *where* C_1 *and* C_2 *are any two paths with the same initial and end points.*

Then we have the following two facts:

(i) $\int_C \nabla f \cdot \mathbf{dr}$ is independent of path. [This is easy enough to prove since all we need to do is look at Theorem 3.2.1. It tells us that in order to evaluate this integral all we need are the initial and end points of the curve. This in turn tells us that the line integral must be independent of path].

(ii) If $\mathbf{F}$ is a conservative vector field then $\int_C \mathbf{F} \cdot \mathbf{dr}$ is independent of path. [This fact is also easy enough to prove. If $\mathbf{F}$ is conservative then it has a potential function, ϕ, and so the line integral becomes $\int_C \nabla \phi \cdot \mathbf{dr}$. Then, using fact (i) we know that this line integral must be independent of path].

Fact (ii) tells us that we can easily evaluate this line integral provided we can find a potential function for $\mathbf{F}$:

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \nabla \phi \cdot \mathbf{dr} = \phi\big(\mathbf{r}(b)\big) - \phi\big(\mathbf{r}(a)\big). \tag{3.11}$$

There are two questions we wish to ask:

1. Given a vector field $\mathbf{F}$ is there any way of determining if it is a conservative vector field?

2. If we know that $\mathbf{F}$ is a conservative vector field how do we go about finding a potential function for the vector field?

The first question is easy to answer because it turns out that to determine if a field is conservative it is sufficient to check if the field is irrotational, that is,

$$\mathbf{F} \text{ conservative} \qquad \Longleftrightarrow \qquad \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}.$$

This follows since a conservative field the vector $\mathbf{F}$ has an associated scalar potential ϕ, that is, $\nabla\phi = \mathbf{F}$. Then from the definition in (2.11) we have curl $\nabla\phi = \nabla \times \nabla\phi = \mathbf{0}$. This can be seen from evaluating the determinant:

$$\nabla \times \nabla\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \mathbf{0}.$$

Now that we know how to identify if a vector field is conservative we need to address how to find a potential function for the vector field. This is actually a fairly simple process. First, we assume that the vector field is conservative and so we know that a potential function ϕ exists such that $\mathbf{F} = \nabla\phi$. If $\mathbf{F} = (P, Q, R)$ we can then say that

$$\nabla\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = (P, Q, R).$$

Or, equating components

$$\frac{\partial \phi}{\partial x} = P, \qquad \frac{\partial \phi}{\partial y} = Q, \qquad \frac{\partial \phi}{\partial z} = R.$$

We then integrate each of these with respect to the appropriate variable. It is usually best to see how to find a potential function in practice from an example or two.

Example 3.2.4 *Show that the vector field* $\mathbf{v} = -g\mathbf{k}$ *(where g is the constant acceleration due to gravity) is conservative. Determine an an associated scalar potential ϕ for the vector field* $\mathbf{v}$*.*

Example 3.2.5 (Old exam question) *Show that* $\mathbf{F} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ *is a conservative vector field and determine an an associated scalar potential ϕ for this vector field. Hence find* $\int_C \mathbf{F}\cdot\mathbf{dr}$ *where C is the curve described by* $\mathbf{r}(t) = (t^2 + 1, 2t^2, t^3)$*, for* $1 \le t \le 2$*.*

Solution. Now

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xy + z^3 & 3xz^2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xy + z^3 & x^2 \end{vmatrix} \\ &= \mathbf{i}(0) - \mathbf{j}(3z^2 - 3z^2) + \mathbf{k}(2x - 2x) = \mathbf{0}. \end{aligned}$$

So $\mathbf{F}$ is conservative and there exists ϕ such that $\mathbf{F} = \nabla\phi$ or

$$\text{(i)} \quad \frac{\partial \phi}{\partial x} = 2xy + z^3, \qquad \text{(ii)} \quad \frac{\partial \phi}{\partial y} = x^2, \qquad \text{(iii)} \quad \frac{\partial \phi}{\partial z} = 3xz^2.$$

Integrating (i) w.r.t. x gives

$$\phi = x^2y + z^3x + f(y,z).$$

Now differentiating this w.r.t. y gives

$$\frac{\partial \phi}{\partial y} = x^2 + \frac{\partial f}{\partial y}.$$

From comparing with (ii) we deduce that $\frac{\partial f}{\partial y} = 0$ and integrating w.r.t. y then gives $f = f(z)$. Hence ϕ becomes

$$\phi = x^2y + z^3x + f(z).$$

Now differentiating this w.r.t. z gives

$$\frac{\partial \phi}{\partial z} = 3z^2x + \frac{\partial f}{\partial z}.$$

From comparing with (iii) we deduce that $\frac{\partial f}{\partial z} = 0$ and so $f = c = \text{const}$. Hence

$$\phi = x^2y + z^3x + c.$$

Finally, to find $\int_C \mathbf{F} \cdot \mathbf{dr}$ we use Theorem 3.2.1, no integration is required! Now, $\mathbf{r}(a) = \mathbf{r}(1) = (2,2,1)$ and $\mathbf{r}(b) = \mathbf{r}(2) = (5,8,8)$. Hence

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \nabla\phi \cdot \mathbf{dr} &= \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)) \\ &= \big[(5)^2(8) + (8)^3(5) + c\big] - \big[(2)^2(2) + (1)^3(2) + c\big] = 2570.\end{aligned}$$

Note that the unknown constant c conveniently cancelled out in this calculation.

3.2.3 Vector Line Integrals

Another common type of line integral of a vector field called a **vector line integral** takes the form $\int_C \mathbf{f}\, ds$ and is defined as follows:

$$\int_C \mathbf{f} ds = \mathbf{i} \int_C f_1 ds + \mathbf{j} \int_C f_2\, ds + \mathbf{k} \int_C f_3\, ds,$$

where $\mathbf{f} = (f_1, f_2, f_3)$. This integral results in a **vector** and to carry integration involves computing three scalar line integrals (which we saw how to deal with in § 3.1).

3.3 Repeated Integrals

An integral of the form:

$$\int_a^b \int_{p(x)}^{q(x)} f(x,y)\,dy\,dx,$$

where a, b are known constants, $p(x)$, $q(x)$ are known functions of x, and $f(x,y)$ is a known function of (x,y) is called a **repeated integral**. It is evaluated by first calculating the **inner** integral:

$$\int_{p(x)}^{q(x)} f(x,y)\,dy,$$

while holding x constant in the integration, i.e. carrying out this integration as if x were a constant. When this integral has been evaluated and the limit values filled in, what remains is a function of x only $= I(x)$ as wherever y appears it has been replaced by $p(x)$, $q(x)$. Now we have:

$$\int_a^b \int_{p(x)}^{q(x)} f(x,y)\,dy\,dx = \int_a^b \left\{ \int_{p(x)}^{q(x)} f(x,y)\,dy \right\} dx = \int_a^b I(x) dx,$$

and evaluation of the repeated integral has reduced to evaluation of a definite integral as $I(x)$ is a known function of x.

General Rule of Thumb:

The limits on the inner integration can be functions of the outer variable (x above) but the limits on the outermost integral must be constants. In the special case when both sets of limits are constants the order of integration may be reversed, i.e.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy,$$

provided a, b, c, d are **constants**.

Example 3.3.1 *Evaluate*

$$I = \int_0^1 \int_{x/2}^{x} xy\,dy\,dx.$$

Example 3.3.2 *Evaluate*

$$I = \int_0^{\pi/4} \int_0^{y} \frac{\sin y}{y}\,dx\,dy.$$

3.4 Double and area integrals

Let $f(x,y)$ be a scalar function of two variables defined over a closed region R in xy space as shown in Figure 3.4. We subdivide R by drawing parallel lines to the x and y axes and number those rectangles which are within R from 1 to n. In each such rectangle we choose a point, say (x_k, y_k) in the kth rectangle, and define:

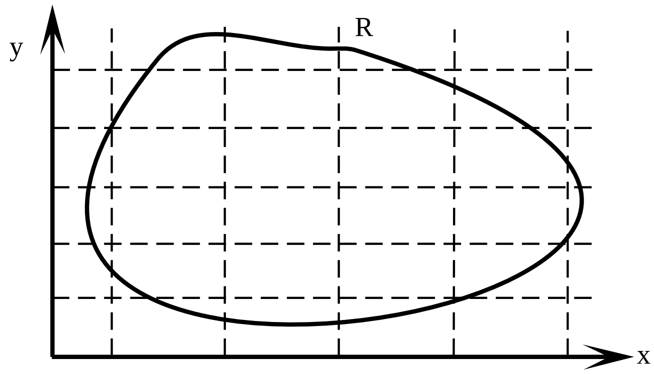

Figure 3.4: *Integral over a 2D area R.*

$$J_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k,$$

where ΔA_k is the area of the kth rectangle. Now let n tend to infinity:

$$\lim_{n\to\infty} J_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \int_R f\, dA = \iint_R f\, dx\, dy,$$

where the limits of the integration in terms of x and y are consistent with the region R. Thus we write the area integral as a **repeated integral** which we can evaluate as in § 3.3. The main difficulty in evaluating area integrals is in choosing the limits of integration to correspond to the region of integration R. Note, the notation $\iint_R f\, dA$ instead of $\int_R f\, dA$ can also be used.

Example 3.4.1 *Evaluate* $\int_R xy\, dA$ *where* R *is the square formed by the points* $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Example 3.4.2 *Evaluate* $\int_R xy\, dA$ *where* R *is bounded by the coordinate planes and the lines* $y = x/2$ *and* $y = 1/2$.

Solution. The region of integration is clearly that in Figure 3.5.

Let us choose to integrate first w.r.t. x and then w.r.t. y so the x variable is the inner variable and during the first integration y is held constant. In this case the region R is described by the inequalities: $0 \le x \le 2y$; $0 \le y \le 1/2$ and the area integral is

$$\begin{aligned} I = \int_R xy\, dA &= \int_{y=0}^{y=1/2} \int_{x=0}^{x=2y} xy\, dx\, dy \\ &= \int_{y=0}^{y=1/2} \left[\frac{x^2 y}{2}\right]_{x=0}^{x=2y} dy \\ &= \int_0^{1/2} 2y^3\, dy = \left[\frac{y^4}{2}\right]_0^{1/2} = \frac{1}{32}. \end{aligned}$$

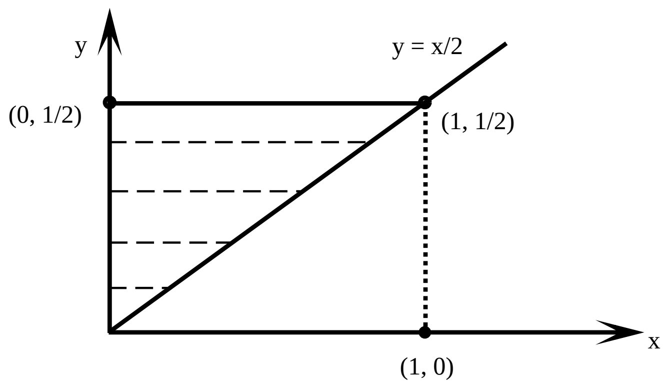

Figure 3.5: *Region of integration for Example 3.4.2.*

Note again that in performing the integration w.r.t. x first we are integrating across horizontal strips and the limits on x vary (as functions of y from strip to strip, see Figure 3.5).

Alternatively, we can reverse the order of integration noting that in this case the region R is described by the inequalities: $0 \leq x \leq 1$; $x/2 \leq y \leq 1/2$. In this case the integral becomes

$$\begin{aligned} I = \int_R xy\,dA &= \int_{x=0}^{x=1} \int_{y=x/2}^{y=1/2} xy\,dy\,dx \\ &= \int_{x=0}^{x=1} \left[\frac{xy^2}{2}\right]_{y=x/2}^{y=1/2} dx \\ &= \int_0^1 \left(\frac{x}{8} - \frac{x^3}{8}\right) dx = \left[\frac{x^2}{16} - \frac{x^4}{32}\right]_0^1 = \frac{1}{32}, \end{aligned}$$

as above.

Example 3.4.3 *Let R be the region that lies under the graph of $y = x^2$ for $0 \leq x \leq 1$. Evaluate $\int_R x\,dA$.*

Example 3.4.4 *Let G be the plane region in the first quadrant bounded by $y = x^2$, $y = 4$ and the y-axis. Find $\iint_G x^2 y\,dx\,dy$.*

Solution. The region of integration is clearly that in Figure 3.6. Thus

$$\begin{aligned} \iint_G x^2 y\,dx\,dy &= \int_0^2 \int_{x^2}^4 x^2 y\,dy\,dx \\ &= \int_0^2 \left[\frac{x^2y^2}{2}\right]_{x^2}^4 dx = \int_0^2 \left(8x^2 - \frac{1}{2}x^6\right) dx = \frac{256}{21}. \end{aligned}$$

Alternatively, we could have

$$\begin{aligned} \iint_G x^2 y\,dx\,dy &= \int_0^4 \int_0^{\sqrt{y}} x^2 y\,dx\,dy \\ &= \int_0^4 \left[\frac{x^3 y}{3}\right]_0^{\sqrt{y}} dy = \frac{1}{3}\int_0^4 y^{5/2}\,dy = \frac{1}{3}\left[\frac{2}{7}y^{7/2}\right]_0^4 = \frac{256}{21}. \end{aligned}$$

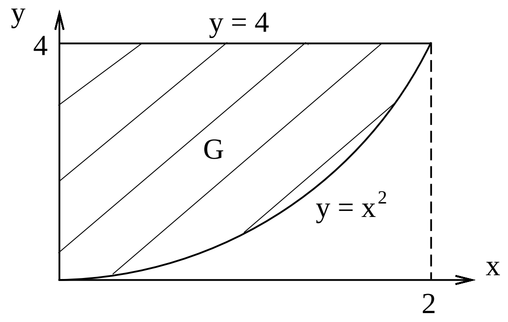

Figure 3.6: *Region of integration for Example 3.4.4*

3.4.1 Geometrical interpretation of double integrals

If $f = f(x, y)$ the equation $z = f(x, y)$ represents a surface in the xyz space. Then $\iint_R f\, dx\, dy$ calculates the **volume** contained between the surface $z = f(x, y)$ and the region R in the xy-plane (see Figure 3.7).

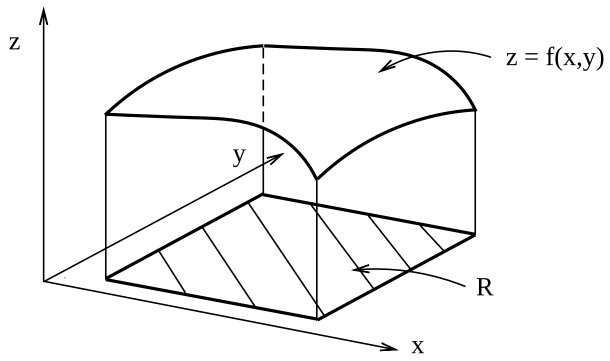

Figure 3.7: *Volume between surface $z = f(x, y)$ and region R in the xy-plane.*

Note the important special case where we set $f(x, y) = 1$ so the area integral becomes

$$\int_R 1\, dA = \iint_R 1\, dx\, dy.$$

This give the area of a region R, i.e.

$$\text{Area of } R = \iint_R 1\, dx\, dy.$$

3.4.2 Change of Variables

With ordinary definite integrals, we are familar with changes of variable. Consider

$$I = \int_0^1 (1 - x^2)^{1/2}\, dx.$$

To evaluate this integral we let $x = \sin\theta$ so $dx = \cos\theta\, d\theta$ and we must also adjust the limits correspondingly. Thus the limit $x = 1$ corresponds to $\theta = \pi/2$ and the limit $x = 0$ corresponds to

$\theta = 0$. Thus

$$I = \int_0^{\pi/2} (1-\sin^2\theta)^{1/2}\cos\theta\, d\theta = \int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{\pi}{4}.$$

Thus in general if $I = \int_a^b f(x)\, dx$ and we wish to carry out a transformation of independent variable $x = x(\theta)$, we write:

$$I = \int_\alpha^\beta f(x(\theta))\frac{dx}{d\theta}\, d\theta,$$

where $x(\alpha) = a$ and $x(\beta) = b$. (Compare with the example above).

For double integrals it is often convenient to change co-ordinate systems (i.e. transform variables). Suppose that we have the integral:

$$I = \iint_R f(x,y)\, dx\, dy,$$

and we wish to transform to new variables u and v such that $x = x(u,v)$ and $y = y(u,v)$, then it can be shown that (though we will not do so here)

$$\iint_R f(x,y)dxdy = \iint_G f(x(u,v), y(u,v))|J(u,v)|\, du\, dv, \tag{3.12}$$

where G is the region in the (u,v) plane corresponding to R in the xy-plane (just as we had to change the limits in the definite integral above from being in terms of x to being in terms of θ) and J is the *Jacobian* determinant defined as

$$J(u,v) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}. \tag{3.13}$$

This is a generalisation (for the case of two independent variables) of the $\frac{dx}{d\theta}$ which arises during the transformation of the definite integral.

To summarise, if we wish to transform a double integral with independent variables (x,y) to new independent variables (u,v) where $x = x(u,v)$ and $y = y(u,v)$, i.e. if we wish to transform $\iint_R f(x,y)\, dx\, dy$ then we must:

(i) Change the limits of the integral correspondingly;

(ii) Write each of x and y in $f(x,y)$ explicitly as a function of (u,v);

(iii) Transform $dx\, dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$. (Note that a rough mnemonic for carrying out this last step is to "imagine" that the $\partial(u,v)$ cancels out the $du\, dv$ in a similar way to the fact that $dx = \frac{dx}{d\theta}\, d\theta$).

We also note that the Jacobian can be used to get the differential element of area in any co-ordinate system, i.e.

$$dA = dx\, dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

The most common transformation in 2D is from cartesian (x, y) co-ordinates to plane polars (r, θ) (which is typically useful if the 2D areas which we are dealing with are circular or partly circular). We set

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and so the Jacobian in (3.13) becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Then from (3.12) we have

$$\iint_R f(x, y)\, dA = \iint_G f(r\cos\theta, r\sin\theta) r\, dr\, d\theta. \tag{3.14}$$

Example 3.4.5 *Evaluate* $I = \iint_R y^2\, dx\, dy$ *by transforming to polar co-ordinates where* R *is the quarter circle* $0 \le y \le (1 - x^2)^{1/2}$, $0 \le x \le 1$.

Solution. It is useful to get an idea of what the region of integration looks like in both systems, i.e. in terms of (x, y) and (r, θ). Recall that plane polar co-ordinates are defined as $x = r\cos\theta$, $y = r\sin\theta$. As we are dealing with a quarter circle in the first quadrant we have the following domains:

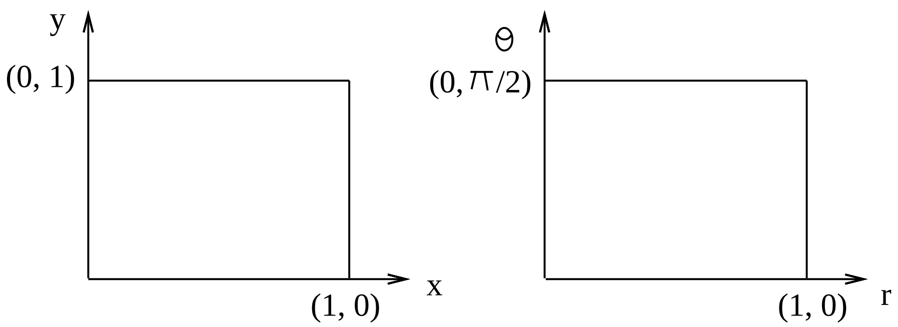

Figure 3.8: *Conversion from rectangular to polar co-ordinates.*

Recall that we need to:

(i) Find new limits for (r, θ);

(ii) Write x and y in terms of (r, θ) in the expression $f(x, y) = y^2$;

(iii) Transform $dx\, dy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\, d\theta$.

We first derive an explicit expression for J for plane polar co-ordinates using (3.13) with $u = r$, $v = \theta$ and recalling that $x = r\cos\theta$, $y = r\sin\theta$ and $J = r$. Considering R in the (x, y) quadrant (see Figure 3.8) we find by inspection that it will be covered in the (r, θ) space if $0 \le r \le 1$ and $0 \le \theta \le \pi/2$. Thus using (3.14)

$$
\begin{aligned}
I = \iint_R y^2\, dx\, dy &= \int_0^{\pi/2} \int_0^1 r^2 \sin^2\theta |J|\, dr\, d\theta \\
&= \int_0^{\pi/2} \int_0^1 r^3 \sin^2\theta\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{4} \sin^2\theta\, d\theta = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta = \frac{\pi}{16}.
\end{aligned}
$$

Note that in transforming to polar co-ordinates, the limits of the integration were simpler than in the original case with cartesians (essentially because polar co-ordinates are better suited for circular shaped domains).

Example 3.4.6 *Find the area of a quarter circle of radius a using polar co-ordinates.*

Exercise: Perform the integration directly (using x and y as independent variables).

3.5 Triple and volume integrals

We can define triple and volume integrals analogously to double integrals. Suppose $f(x, y, z)$ is defined in some *volume* of space V. Then we break V up into small parallelepipeds, each of volume ΔV, and we form the sum:

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta V_k.$$

Then as $n \to \infty$ we define

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta V_k = \int_V f\, dV = \iiint_V f(x, y, z)\, dx\, dy\, dz,$$

to be the volume integral of $f(x, y, z)$ over the region. Like area integrals, volume integrals are easy because the differential element of volume $dV = dx\, dy\, dz$. Also, the notation $\iiint_V f\, dV$ is often used instead of $\int_V f\, dV$.

3.5.1 Geometric interpretation

By analogy with double integrals, $\iiint_V f\, dx\, dy\, dz$ is the "volume" of the 4D object above the 3D "plane" described by V (i.e. one higher dimension than that shown in Figure 3.7).
However, setting $f(x, y, z) \equiv 1$ gives an integral $\int_V 1\, dV$ which equals the volume of the region V.
If a liquid occupies the volume V with density $\rho = \rho(x, y, z)$ then the total **mass** in V is

$$\text{mass} = \int_V \rho\, dV = \iiint_V \rho(x, y, z)\, dx\, dy\, dz.$$

If the density ρ is the same at every point in the body, i.e. $\rho =$ constant, this integral reduces to the familiar result:

$$\text{mass} = \rho \iiint_V dx\,dy\,dz = \rho V \qquad \Longrightarrow \qquad \text{mass} = \text{density} \times \text{volume}.$$

3.5.2 Evaluation of Triple Integrals

This is usually done by evaluation of **repeated integrals**. In this instance we need to generalise for the case of double integrals. We now have to deal with three integrals so we have an inner, a middle and an outer integration. The innermost limits may be functions of two variables (the outer two variables), the middle limits may be functions of a single variable (the remaining outer variable) and the outermost limits will be constants, e.g. a typical triple integral looks like:

$$\int_{z=e}^{z=f} \int_{y=c(z)}^{y=d(z)} \int_{x=a(y,z)}^{x=b(y,z)} f(x,y,z)\,dx\,dy\,dz.$$

Example 3.5.1 *Evaluate* $\int_V (y^2 + z^2)\,dV$ *if* V *is the three dimensional volume defined by* $|x| \le a$, $|y| \le b$, $|z| \le c$ *where* a, b, c *are known constants.*

3.5.3 Changes of variables in triple integrals

Analogous to the 2D situation, if we are evaluating the volume integral $\int_V f(x,y,z)\,dV$ and we wish to transform from independent variables (x,y,z) to (u,v,w) with $x = x(u,v,z)$, $y = y(u,v,w)$ and $z = z(u,v,w)$, then

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(u,v,w)\,|J|\,du\,dv\,dw$$

where J is the Jacobian determinant defined by

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix},$$

and V' is the region in the (u,v,w) space corresponding to V in the xyz-space.
Again we transform:

(i) Limits of integration;

(ii) Each (x,y,z) in $f(x,y,z)$ is written in terms of (u,v,w) (giving essentially a new function $g(u,v,w)$);

(iii) The differential volume element $dV = dx\,dy\,dz = |J|\,du\,dv\,dw$.

We now give some very useful co-ordinate transformations in 3D.

Cylindrical co-ordinates

To transform from cartesian co-ordinates (x, y, z) to cylindrical co-ordinates (r, θ, z) we set

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \tag{3.15}$$

as shown in Figure 3.9(a). Then

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

So, in cylindrical polar co-ordinates we have

$$dV = r\,dr\,d\theta\,dz.$$

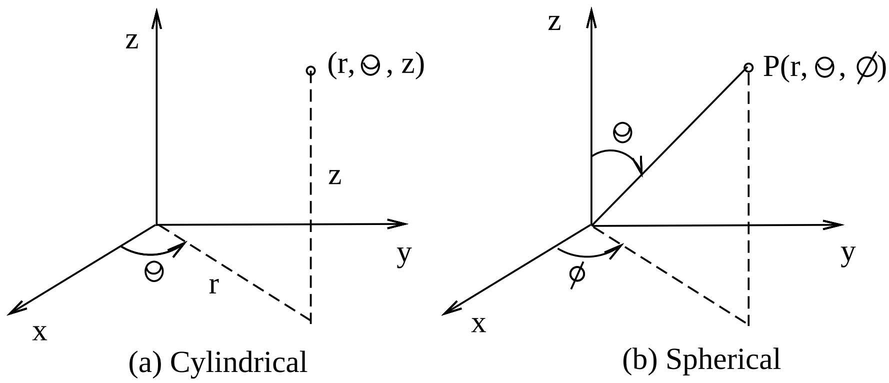

Figure 3.9: *(a) Cylindrical co-ordinates, (b) Spherical co-ordinates.*

Spherical co-ordinates

Let P have rectangular cartesian co-ordinates (x, y, z), let r denote the distance from the origin to the point P, let θ be the angle between OP and the positive z-axis (and so $0 \le \theta \le \pi$) and ϕ be the angle between the xz-plane and the plane containing P and the z-axis (and so $0 \le \phi \le 2\pi$), see Figure 3.9(b). [**Note**: θ is defined differently here to the θ in cylindrical co-ordinates].

To transform from cartesian co-ordinates (x, y, z) to spherical co-ordinates (r, θ, ϕ) we therefore set

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta. \tag{3.16}$$

Exercise: Show that for a transformation from cartesian to spherical coordinates

$$J = r^2\sin\theta. \tag{3.17}$$

Example 3.5.2 *Calculate the volume of a sphere using spherical coordinates.*

Example 3.5.3 *Evaluate* $\iint_V 16z\,dV$ *where* V *is the upper half of the sphere* $x^2+y^2+z^2=1$.

3.6 Surfaces

Volume and area integrals were "easy" in the sense that the differential elements of area and volume were $dA = dx\,dy$ and $dV = dx\,dy\,dz$ respectively and we could easily reduce these to a repeated integral. Consider an integral defined on the surface of a sphere, where $f(x,y,z)$ is defined at every point in space and in particular on the sphere. In the usual way we can define an integral in terms of a limiting summation:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k,y_k,z_k)\Delta S_k = \iint_S f(x,y,z)dS,$$

where we break the surface S into little elements ΔS_k and (x_k,y_k,z_k) is located in the middle of each element.

The general surface integral is of the form $\int_S f\,dS$ where dS corresponds to a differential element of surface on the sphere, but how do we integrate this? In fact, surface integrals are analogous to line integrals, and the strategy then was to get a **parametric description of the curve**, and use that to simplify the expression.

Let a variable point P have a position vector $\mathbf{r}(u,v) = (x(u,v),y(u,v),z(u,v))$ where (u,v) are parameters in some continuous region of the uv-plane and (x,y,z) are single valued functions of u, v. If u is held constant and v is allowed to vary we have essentially one parameter and we get a family of **curves** called the v co-ordinate curves. Similarly we have the family of u co-ordinate curves by holding v constant and varying u. The network of all such curves describes the surface S.

Example 3.6.1 *The position vector*

$$\mathbf{r} = \mathbf{r}(\theta,z) = (\cos\theta,\sin\theta,z),$$

with $0\le\theta\le 2\pi$, $0\le z\le 1$ *defines the surface of the cylinder in* (x,y,z) *space. If we hold* z *constant then* $x^2+y^2=\cos^2\theta+\sin^2\theta=1$ *describes a circle in the plane* $z = const$. *Now letting* z *vary from 0 to 1 picks out all the circles between* $z=0$ *and* $z=1$, *i.e. in totality we pick out a cylindrical surface in space (see Figure 3.10).*

In this example $u=\theta$ *and* $v=z$ *and so* θ *and* z *are the parameters in the parametric description of the surface. Note that parametric definitions of cylinders and spheres can be obtained from the definitions of these co-ordinate systems. For example, in cylindricals we have* $x=r\cos\theta$, $y=r\sin\theta$,

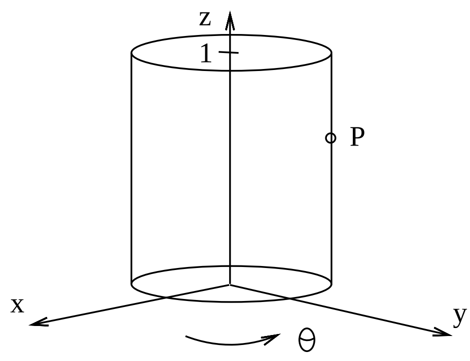

Figure 3.10: *Surface of a cylinder.*

$z = z$. *To obtain the equation of a surface we require only* **two** *parameters running independently. In the example above* r *does not run freely but is fixed at the value* $r = 1$ *(a known constant) and* θ, z *are the parameters with restrictions on their values. Hence we obtain the parametric representation* $x = \cos\theta$, $y = \sin\theta$, $z = z$.

3.6.1 Open and Closed Surfaces

A surface S is **open** if every two points not lying on S can be joined by a continuous curve which does not cross S. A surface S is **closed** if it divides a space into two regions, R_1 and R_2 say, such that every continuous curve joining a point in R_1 to a point in R_2 crosses S at least once. For example, the cap of a sphere is open but a complete spherical shell is closed. The cap of a sphere can be closed off with a circular plane (see Figure 3.11).

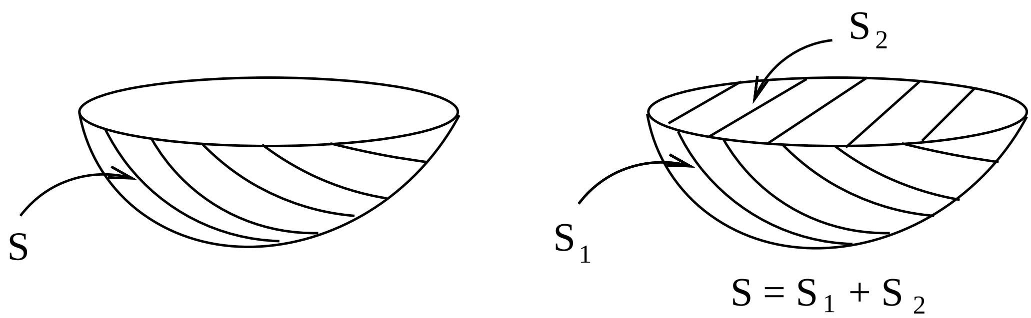

Figure 3.11: *An open surface (spherical cap); the same cap closed off.*

3.6.2 Unit Normal Vector

Consider a surface S defined parametrically by $\mathbf{r} = \mathbf{r}(u, v)$. At any point on S, the vectors $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ (denoted for convenience by $\mathbf{r}_u$ and $\mathbf{r}_v$) are **tangential** to the u and v co-ordinate curves. This is because, for example, the u co-ordinate curves are defined by keeping v constant and allowing u

to vary. So $\mathbf{r} = \mathbf{r}(u)$ which has just one parameter so it describes a curve in space. From Chapter 1, $\frac{\partial \mathbf{r}}{\partial u}$ describes the tangent to the curve at any point, see Figure 3.12.

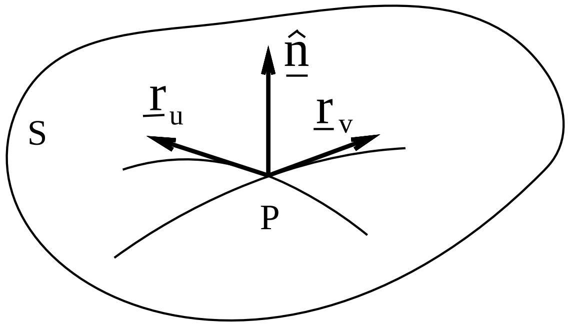

Figure 3.12: *The unit normal to a surface S.*

The **unit normal vector** to the surface S is therefore given by

$$\hat{\mathbf{n}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}, \tag{3.18}$$

since from the definition of the vector (or cross) product this is a vector which is perpendicular to both $\mathbf{r}_u$ and $\mathbf{r}_v$. Recall that $\mathbf{r}_u \times \mathbf{r}_v = -\mathbf{r}_v \times \mathbf{r}_u$ and so interchanging $\mathbf{r}_u$ and $\mathbf{r}_v$ gives a unit vector but in the opposite direction. It is usual to label one side of a surface as "outer" and to have the normal vector pointing in this direction, and to refer to it as the outward normal. For a closed surface (e.g. a spherical shell) the outer direction is that pointing "outwards" (i.e. from the interior to the exterior).

Example 3.6.2 *Consider the cylindrical surface given parametrically by* $\mathbf{r}(\theta, z) = (\cos\theta, \sin\theta, z)$ *and find the unit normal at any point to the surface.*

3.6.3 Surface area

Let $P_0(u, v)$ be a point on the surface S described by $\mathbf{r} = \mathbf{r}(u, v)$. Let $P_1(u+du, v)$, $P_2(u+du, v+dv)$, and $P_3(u, v+dv)$ be neighbouring points on either the u or v co-ordinate curves, as in Figure 3.13. $P_0\, P_1\, P_2\, P_3$ is approximately a parallelogram so the area of the surface element is approximately

$$dS \approx \left|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_3}\right|.$$

Now, since $\mathbf{r}_u$ and $\mathbf{r}_v$ lie parallel to $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_3}$ respectively (see Figure (3.12)) we can say

$$\overrightarrow{P_0P_1} \approx \frac{\partial \mathbf{r}}{\partial u} du, \qquad \overrightarrow{P_0P_3} \approx \frac{\partial \mathbf{r}}{\partial v} dv.$$

Thus in the limit as du, $dv \to 0$ the above approximations become exact and we are dealing with a differential element of surface. Thus

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv\right| = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv = \left|\mathbf{r}_u \times \mathbf{r}_v\right| du\, dv. \tag{3.19}$$

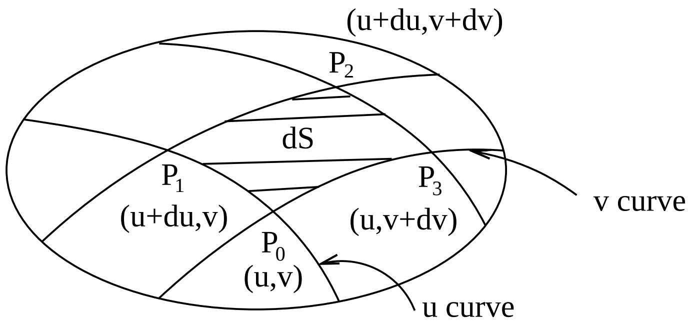

Figure 3.13: *A surface element dS.*

With the above derivation as motivation we define surface area as follows:

Definition 3.6.1 *Given a surface S defined parametrically by $\mathbf{r} = \mathbf{r}(u, v)$ the surface area of S is*

$$\iint_S dS = \iint_S |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv, \tag{3.20}$$

where the ranges of u and v are such that the whole of S is covered.

Example 3.6.3 *Calculate the surface area of the cylindrical surface defined by $\mathbf{r}(\theta, z) = (a\cos\theta, a\sin\theta, z)$, if $0 \le \theta \le 2\pi$, $0 \le z \le b$, where a and b are known constants.*

3.7 Surface Integrals

Let S be a surface with $\mathbf{r} = \mathbf{r}(u, v)$. Let R denote the region in the (u, v) space corresponding to points on S. If a scalar field $\Omega(x, y, z)$ and a vector field $\mathbf{f}(z, y, z)$ are defined at all points on S, then $\mathbf{r}(x, y, z) = (x(u, v), y(u, v), z(u, v))$ and so on S we have $\Omega = \Omega(u, v)$ and $\mathbf{f} = \mathbf{f}(u, v)$.
We define the surface integrals of Ω and $\mathbf{f}$ over S as follows

$$\iint_S \Omega \, dS = \iint_R \Omega(u, v) \, |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv \tag{3.21}$$

$$\iint_S \mathbf{f} \cdot \mathbf{dS} = \iint_S \mathbf{f} \cdot \hat{\mathbf{n}} \, dS = \iint_R \mathbf{f}(u, v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv. \tag{3.22}$$

Remarks:

- By writing $\Omega(u, v)$ and $\mathbf{f}(u, v)$ we really mean $\Omega(x(u, v), y(u, v), z(u, v))$ and $\mathbf{f}(x(u, v), y(u, v), z(u, v))$.
- dS is interpreted as an element of surface area, and the vector $\mathbf{dS}$ is defined to be $\hat{\mathbf{n}} \, dS$ where dS is the (scalar) element of surface area. Thus (3.22) can be derived from (3.21) from substituting $\hat{\mathbf{n}} = (\mathbf{r}_u \times \mathbf{r}_v)/|\mathbf{r}_u \times \mathbf{r}_v|$ from equation (3.18).

- (3.21) is the surface integral of a *scalar field* and (3.22) is the surface integral of a *vector field.*
- Both (3.21) and (3.22) result in **scalar functions**.
- $\int_S \mathbf{f} \cdot \mathbf{dS}$ is sometimes called the **flux** of $\mathbf{f}$.

We define *three* other possible surface integrals. If $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ then

$$\iint_S \mathbf{f}\, dS = \left(\iint_S f_1\, dS\right)\mathbf{i} + \left(\iint_S f_2\, dS\right)\mathbf{j} + \left(\iint_S f_3\, dS\right)\mathbf{k}. \tag{3.23}$$

Also

$$\int_S \Omega\, \mathbf{dS} = \iint_S \Omega\, \hat{\mathbf{n}}\, dS, \qquad \iint_S \mathbf{f} \times \mathbf{dS} = \iint_S \mathbf{f} \times \hat{\mathbf{n}}\, dS. \tag{3.24}$$

All three of these integrals result in **vector functions**.

Example 3.7.1 *Evaluate* $\iint_S \Omega\, dS$ *if*

(i) $\Omega = x^2 + y^2$ *and* S *is the surface* $x^2 + y^2 + z^2 = a^2$*,*

(ii) $\Omega = x^2 + y^2$ *and* S *is the surface of the cube* $|x| \le a$*,* $|y| \le a$*,* $|z| \le a$*,*

where a *is a known constant.*

Solution (i). The parametric equations for the sphere are

$$\mathbf{r} = \mathbf{r}(u, v), \qquad x = a \sin u \cos v, \qquad y = a \sin u \sin v, \qquad z = a \cos u,$$

where a is constant and (u, v) are parameters: $0 \le u \le \pi$, $0 \le v \le 2\pi$. Of course (u, v) are just (θ, ϕ) of the spherical co-ordinate system (r, θ, ϕ) which is ideal for mapping out the surface of a sphere. In sphericals the surface of a sphere of radius a is given by simply fixing $r = a$, the radius of the sphere and using transformation equations from Chapter 2 to write (x, y, z) in terms of the two remaining parameters (θ, ϕ) (as r is constant it does not act as a parameter). This is similar to parameterising the surface of a cylinder (discussed at the beginning of Section § 3.6): there we used cylindrical co-ordinates but kept r fixed to obtain only two independent parameters, θ, z.
Note that we are **not** changing co-ordinate system, i.e. we are still using cartesian co-ordinates (x, y, z) but we are using our experience with spherical co-ordinates to obtain a parametric equation for the sphere in terms of cartesian co-ordinates.
On S, $\Omega = x^2 + y^2 = a^2 \sin^2 u$, while the partial derivatives of $\mathbf{r}$ w.r.t. u, v are

$$\mathbf{r}_u = (a \cos u \cos v, a \cos u \sin v, -a \sin u), \qquad \mathbf{r}_v = (-a \sin u \sin v, a \sin u \cos v, 0).$$

Thus (check)

$$\mathbf{r}_u \times \mathbf{r}_v = (a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u),$$

and so

$$|\mathbf{r}_u \times \mathbf{r}_v| = a^2 \sin u.$$

Hence using (3.21) we have

$$\iint_S \Omega \, dS = \int_0^{2\pi} \int_0^{\pi} \Omega(u,v)|\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \int_0^{2\pi} \int_0^{\pi} a^4 \sin^3 u \, du \, dv$$
$$= a^4 \int_0^{2\pi} \Big[-\cos u + \frac{1}{3}\cos^3 u \Big]_0^{2\pi} dv = \frac{8}{2}\pi a^4.$$

Note: in evaluating the above integral we used the fact that

$$\int \sin^3 u \, du = -\cos u + \frac{1}{3}\cos^3 u.$$

Solution (ii). To evaluate the second integral we do so by considering the six faces of the cube (i.e. $x = \pm a$, $y = \pm a$, $z = \pm a$).

On the two faces $z = \pm a$ we know that each face is parallel to the xy-plane and hence the element of surface $dS = dx\,dy$. Define:

$$C_1 = \text{integral over face } z = \pm a = 2 \int_{-a}^{a} \int_{-a}^{a} (x^2 + y^2) \, dx \, dy = \frac{16a^4}{3}.$$

On the two faces $x = \pm a$ we know that each face is parallel to the yz-plane and hence the element of surface $dS = dy\,dz$. Define:

$$C_2 = \text{integral over face } x = \pm a = 2 \int_{-a}^{a} \int_{-a}^{a} (a^2 + y^2) \, dy \, dz = \frac{32a^4}{3}.$$

On the two faces $y = \pm a$ we know that each face is parallel to the xz-plane and hence the element of surface $dS = dx\,dz$. Define:

$$C_3 = \text{integral over face } y = \pm a = 2 \int_{-a}^{a} \int_{-a}^{a} (x^2 + a^2) \, dx \, dz = \frac{32a^4}{3}.$$

Hence

$$\iint_S \Omega \, dS = C_1 + C_2 + C_3 = \frac{80a^4}{3}.$$

Example 3.7.2 *Let* $\mathbf{r}$ *be the position vector of a point* P*. Evaluate* $\int_S \mathbf{r} \cdot \mathbf{dS}$ *where* S *is the surface of the paraboloid* $z = 2 - x^2 - y^2$ **above** *the* xy *plane (i.e.* $z = 0$*), see Figure 3.14.*

Solution. S is given parametrically by $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) = (u, v, 2 - u^2 - v^2)$ by inspection. We also need to restrict u and v so that $z = 0$. Now

$$\mathbf{r}_u = (1, 0, -2u), \qquad \mathbf{r}_v = (0, 1, -2v),$$

and so

$$\mathbf{r}_u \times \mathbf{r}_v = (2u, 2v, 1).$$

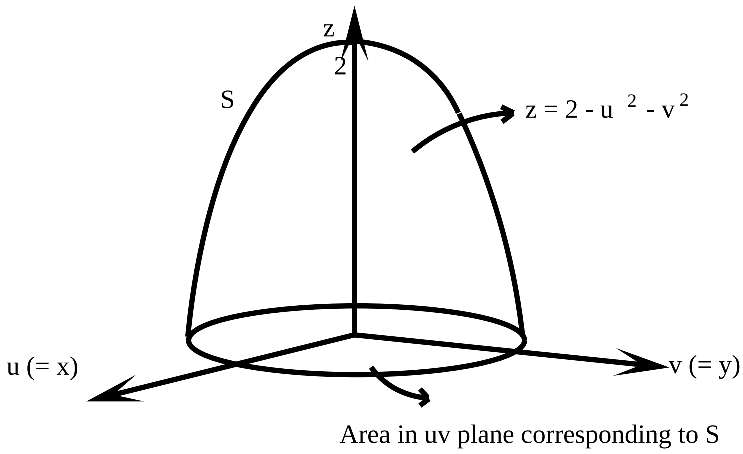

Figure 3.14: *Paraboloid $z = 2 - x^2 - y^2$ above the xy plane.*

On S, $\mathbf{r} = (u, v, 2 - u^2 - v^2)$ and so

$$\mathbf{r} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = (u, v, 2 - u^2 - v^2) \cdot (2u, 2v, 1) = 2 + u^2 + v^2.$$

Thus

$$\iint_S \mathbf{r} \cdot \mathbf{dS} = \iint_R \mathbf{r} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv = \iint_R (2 + u^2 + v^2)\, du\, dv. \tag{3.25}$$

We need to find limits for (u, v) which correspond to picking out the surface S in (x, y, z) space (i.e. the region R). The surface is defined parametrically by: $\mathbf{r} = (u, v, 2 - u^2 - v^2)$ and we know that $z \geq 0$, i.e. $2 - u^2 - v^2 \geq 0$.

In the (u, v) plane (which is the same as the (x, y) plane), the intersection of the surface S with this plane is given by $2 - u^2 - v^2 = 0$ which implies $u^2 + v^2 = 2$, i.e. the intersection of S with the (u, v) plane is a circle with radius $\sqrt{2}$ and this is the area over which we wish to carry out the integration. Now choosing u to be the **outer** variable during the integration means u must be restricted: $-\sqrt{2} \leq u \leq \sqrt{2}$. The inner variable is more difficult as it is a function of the outer variable and clearly we require $-\sqrt{2 - u^2} \leq v \leq \sqrt{2 - u^2}$. Thus the integral reduces to

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-u^2}}^{\sqrt{2-u^2}} (2 + u^2 + v^2)\, du\, dv = 6\pi.$$

Alternatively, we could use the fact that R is the circle $u^2 + v^2 = 2$ and convert to polar co-ordinates:

$$u = r\cos\theta, \qquad v = r\sin\theta,$$

where $0 \leq r \leq \sqrt{2}$, $0 \leq \theta \leq 2\pi$. Then (3.25) becomes (using the transformation in (3.12))

$$\iint_S \mathbf{r} \cdot \mathbf{dS} = \int_0^{2\pi} \int_0^{\sqrt{2}} (2 + r^2) r\, dr\, d\theta = \int_0^{2\pi} \left[r^2 + \frac{r^4}{4} \right]_0^{\sqrt{2}} d\theta = \int_0^{2\pi} 3\, d\theta = 6\pi.$$

3.7.1 Special case for surface integrals of scalar fields

Suppose the surface S is given by $z = g(x, y)$. Then $u = x$ and $v = y$ and so we can write

$$\mathbf{r}(u, v) = \mathbf{r}(x, y) = (x, y, g(x, y)),$$

and (3.21) becomes

$$\iint_S \Omega \, dS = \iint_R \Omega(x, y, g(x, y)) \, |\mathbf{r}_x \times \mathbf{r}_y| \, dx \, dy. \tag{3.26}$$

Now

$$\mathbf{r}_x = (1, 0, g_x), \qquad \mathbf{r}_y = (0, 1, g_y),$$

[where $g_x \equiv \frac{\partial g}{\partial x}$ etc.] and so

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = -g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}.$$

Hence (3.26) reduces to

$$\iint_S \Omega \, dS = \iint_R \Omega(x, y, g(x, y)) \sqrt{1 + (g_x)^2 + (g_y)^2} \, dx \, dy. \tag{3.27}$$

Note that we can derive similar formulas if $x = g(y, z)$ or $y = g(x, z)$.

Example 3.7.3 *Evaluate $\iint_S (xy + z) \, dS$ where S is part of the plane $x + y + z = 2$ that lies in the first octant. (see Figure 3.15).*

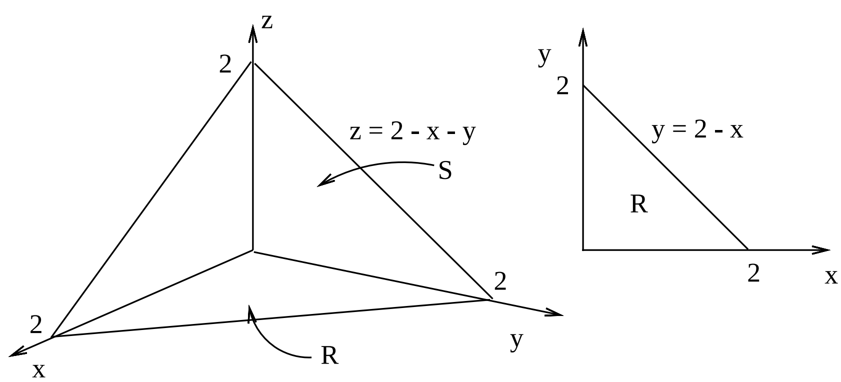

Figure 3.15: *Plane $x + y + z = 2$ lying in the first octant.*

3.8 Integral Theorems

3.8.1 Introduction

The theorems in this section relate certain line integrals to double integrals, certain surface integrals to volume integrals, and certain line integrals to volume integrals. (see Figure 3.16).

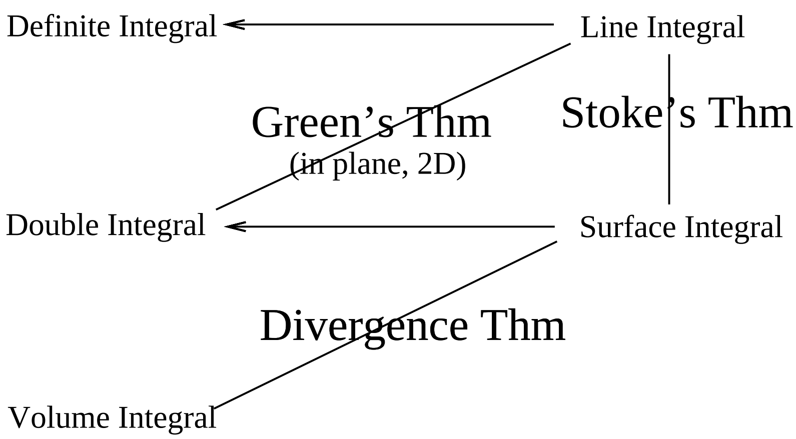

Figure 3.16: *Summary of the relationship between Integral Theorems.*

We will look at three theorems, the Divergence Theorem (Gauss' Theorem), Green's Theorem in two dimensions and Stokes' Theorem.

3.8.2 The Divergence Theorem (Gauss' Theorem)

Theorem 3.8.1 *Consider a* **closed** *region V bounded by a piecewise smooth closed surface S. If the vector* $\mathbf{f}$ *is defined and continuously differentiable throughout V then*

$$\iint_S \mathbf{f}\cdot\mathbf{dS} = \iiint_V \nabla\cdot\mathbf{f}\,dV. \tag{3.28}$$

"The flux of a vector equals the volume of its divergence".

We say that

$$\text{Flux} = \iint_S \mathbf{f}\cdot\mathbf{dS},$$

which can be thought of as the rate of "flow through S".

Intuitive justification of Gauss's Theorem

Consider the case of a compressible liquid, as discussed in § 2.7, where $\mathbf{u}$ is the velocity vector and ρ is the density. If we replace $\mathbf{f}$ with $\rho\mathbf{u}$ (also a vector, and it denotes the mass of fluid) then (3.28) becomes

$$\iint_S (\rho\mathbf{u})\cdot\mathbf{dS} = \iiint_V \nabla\cdot(\rho\mathbf{u})\,dV.$$

Now the LHS is the mass flux of liquid across the surface S, i.e. the mass of liquid passing through the surface S in unit time. Recall (in § 2.7) that we showed $\nabla\cdot(\rho\mathbf{u})\,dx\,dy\,dz = \nabla\cdot(\rho\mathbf{u})\,dV$ represents the net outflow of liquid for any volume element $dx\,dy\,dz$. Thus the integral on the RHS represents the total outflow of liquid out of the volume V.

Hence the divergence theorem can be said to be a statement of conservation of mass (in this instance but it could be any conserved quantity in a particular physical application).

Note: For an incompressible fluid $\rho =$ constant and $\nabla \cdot \mathbf{u} = 0$. Therefore $\iiint_V \nabla \cdot (\rho \mathbf{u})\, dV = 0$ and so the flux of liquid across **any** closed surface in the liquid must also be zero. (If the volume is already full of liquid, you cannot add any more as it is incompressible so whatever flows in one side must simultaneously be flowing out the other side).

Uses of the divergence theorem

The divergence theorem equates a particular volume integral to a particular surface integral. We recall that volume integrals are usually easier to evaluate than surface integrals so if we wish to evaluate a surface integral of the type on the LHS of the divergence theorem statement (3.28), we can avoid doing so and instead evaluate an (equivalent) volume integral. Apart from this the divergence theorem (and all the integral theorems) are very useful for deriving theoretical results.

Example 3.8.1 *Use the divergence theorem to evaluate* $\iint_S \mathbf{f} \cdot \mathbf{dS}$ *where* $\mathbf{f} = (x^3, y^3, z^3)$ *and* S *is the spherical surface* $x^2 + y^2 + z^2 = a^2$.

Solution. We wish to evaluate $I = \iint_S \mathbf{f} \cdot \mathbf{dS}$. According to the divergence theorem this is equivalent to the volume integral $I = \iiint_V \nabla \cdot \mathbf{f}\, dV$ so we evaluate the volume integral instead of evaluating the surface integral.

$$\mathbf{f} = (x^3, y^3, z^3) \qquad \Longrightarrow \qquad \nabla \cdot \mathbf{f} = 3x^2 + 3y^2 + 3z^2,$$

and so

$$I = 3 \iiint_V (x^2 + y^2 + z^2)\, dx\, dy\, dz.$$

As we are dealing with the volume inside a spherical surface it is simplest to change to spherical co-ordinates at this point and use the change of co-ordinate rule (and Jacobian, as in (3.17)) to evaluate this integral. So we transform from (x, y, z) to (r, θ, ϕ) and note that the sphere in sphericals is defined by $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. Recall also the relationship between cartesians and sphericals (i.e. (3.16) which we repeat here for convenience)

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta.$$

Thus the integrand

$$x^2 + y^2 + z^2 = r^2 \sin^2\theta \cos^2\theta + r^2 \sin^2\theta \sin^2\theta + r^2 \cos^2\theta = r^2,$$

and so

$$I = 3 \iiint_V (x^2 + y^2 + z^2)\, dx\, dy\, dz = 3 \iint_{V'} r^2 |J|\, dr\, d\theta\, d\phi.$$

Now $J = r^2 \sin\theta$ and so we have

$$I = 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2(r^2\sin\theta)\,dr\,d\theta\,d\phi = 3\int_0^{2\pi}\int_0^{\pi}\frac{a^5}{5}\sin\theta\,d\theta\,d\phi$$
$$= \frac{3a^5}{5}\int_0^{2\pi}\big[-\cos\theta\big]_{\theta=0}^{\theta=\pi}\,d\phi = \frac{6a^5}{5}\int_0^{2\pi}d\phi = \frac{12a^5\pi}{5}.$$

Exercise: Verify the divergence theorem when $\mathbf{f} = \mathbf{r} = (x, y, z)$ and S is the sphere $x^2+y^2+z^2 = a^2$.

3.8.3 Green's Theorem in the Plane

This relates a double integral over a plane region to a line integral over the boundary.

Theorem 3.8.2 *Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $f_1(x, y)$ and $f_2(x, y)$ be functions which are continuous and have continuous partial derivatives $\partial f_1/\partial y$ and $\partial f_2/\partial x$ everywhere in R. Then:*

$$\iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \oint_C (f_1 dx + f_2 dy). \tag{3.29}$$

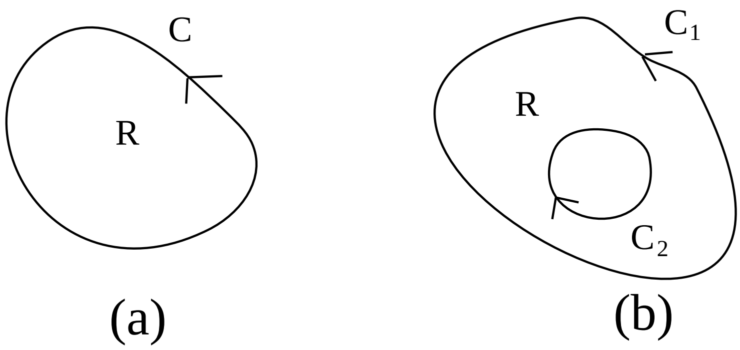

Figure 3.17: *Types of regions to which Green's theorem in the plane can be applied. The region in (b) has a "hole".*

Remarks:

1. The integral is taken along the **entire** boundary C or R such that R is on the left as one advances in the direction of integration (see Figure 3.17(a)). Note that the theorem also applies to more complicated regions like that shown in Figure 3.17(b). This is a region with a hole and it is necessary to bear in mind that the boundary of this region consists of **two** curves along which the line integral must be evaluated.

2. The above formula can be written in vector notation as:

$$\iint_R (\text{curl } \mathbf{f})\cdot\mathbf{k}\,dx\,dy = \oint_C \mathbf{f}\cdot\mathbf{dr}$$

where $\mathbf{f} = (f_1, f_2, 0) = f_1\mathbf{i} + f_2\mathbf{j}$.

Example 3.8.2 *This is a well known example using Green's theorem in the plane to write the area of a plane region as a line integral over the boundary.*

Solution. We know that the area of the plane region R is given by $\iint_R 1\,dx\,dy$. Thus using Green's Theorem (3.29) we want to choose f_1 and f_2 such that

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 1.$$

(i) Let $f_1(x,y) = 0$ and $f_2(x,y) = x$. Then (3.29) says that

$$\iint_R dx\,dy = \oint_C x\,dy. \tag{3.30}$$

Note that the RHS can be written as $\oint_C \mathbf{f}\cdot\mathbf{dr}$ where $\mathbf{f} = (0, x)$.

(ii) Alternatively, let $f_1(x,y) = -y$ and $f_2(x,y) = 0$. Then (3.29) says that

$$\iint_R dx\,dy = -\oint_C y\,dx. \tag{3.31}$$

Note that the RHS can be written as $\oint_C \mathbf{f}\cdot\mathbf{dr}$ where $\mathbf{f} = (-y, 0)$.

(ii) Finally, we can add (3.30) and (3.31) and divide by two to give

$$\iint_R dx\,dy = \frac{1}{2}\oint_C (x\,dy - y\,dx). \tag{3.32}$$

Note that the RHS can be written as $\frac{1}{2}\oint_C \mathbf{f}\cdot\mathbf{dr}$ where $\mathbf{f} = (-y, x)$.

Example 3.8.3 *Use Green's theorem in the plane to find the area of the circle $x^2 + y^2 = a^2$ where a is a known constant.*

Example 3.8.4 *Use Green's theorem to find $\oint_C xy\,dx + x^2y^3\,dy$ where C is the triangle with vertices* $(0,0)$, $(1,0)$, $(1,2)$.

Solution. We use (3.29). Let $f_1 = xy$ and $f_2 = x^2y^3$. Then $\frac{\partial f_1}{\partial y} = x$ and $\frac{\partial f_2}{\partial x} = 2xy^3$ and so

$$\oint_C xy\,dx + x^2y^3\,dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \iint_R \left(2xy^3 - x\right) dx\,dy. \tag{3.33}$$

The region R is given in Figure 3.18 and so, as discussed in § 3.4, if we integrate first w.r.t. y and second w.r.t. x then the region R is described by $0 \le y \le 2x$, $0 \le x \le 1$. Thus (3.33) becomes

$$\oint_C xy\,dx + x^2y^3\,dy = \int_0^1\int_0^{2x} (2xy^3 - x)\,dy\,dx = \int_0^1 \left[\frac{1}{4}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \frac{2}{3}.$$

Note that it was much easier to evaluate the double integral than to integrate the line integral $\oint_C xy\,dx + x^2y^3\,dy$ (which would have involved splitting C into three separate line segments).

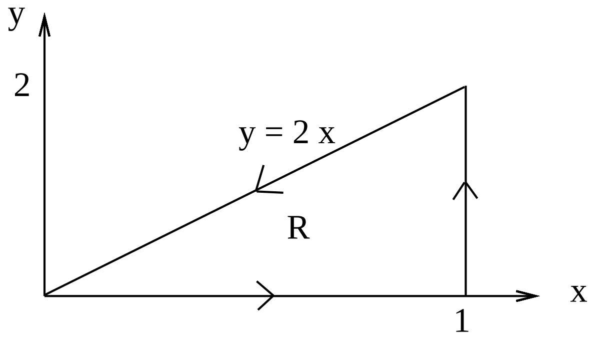

Figure 3.18: *Triangle with vertices* $(0,0)$, $(1,0)$, $(1,2)$.

3.8.4 Stokes' Theorem

This theorem provides a transformation between surface integrals and line integrals.

Theorem 3.8.3 *Let S be a piecewise smooth oriented surface in space bounded by a piecewise smooth curve C. Let $\mathbf{f}(x,y,z)$ have continuous partial derivatives everywhere. Then*

$$\oint_C \mathbf{f}\cdot\mathbf{dr} = \iint_S (\nabla\times\mathbf{f})\cdot\mathbf{dS}, \tag{3.34}$$

with $\mathbf{dS} = \hat{\mathbf{n}}\,dS$ and $\hat{\mathbf{n}}$ being the unit normal to the surface oriented using the right rule (i.e. with the fingers along the curve and the thumb giving the positive direction of the normal, see Figure 3.19).

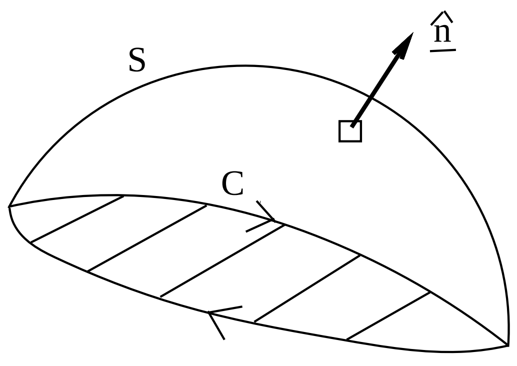

Figure 3.19: *Surface (open, i.e. not including the shaded region) and bounding curve for Stokes' theorem.*

Note that the theorem says: "the circulation of a vector (i.e. the line integral) equals the flux of its curl".

Example 3.8.5 *Verify Stokes' theorem for the vector field $\mathbf{f} = (x^2y, z, 0)$ and the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$ (see Figure 3.20).*

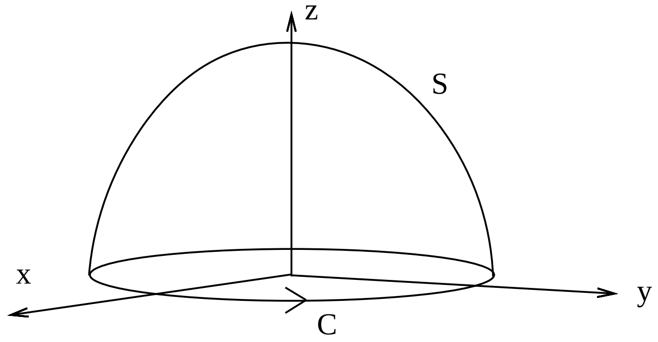

Figure 3.20: *Hemisphere* $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Solution. We obviously need to evaluate both sides of (3.34) and show they are equal.

(i) *Evaluate the line integral*: C is the curve $x^2 + y^2 = a^2$ and is given parametrically by

$$\mathbf{r}(t) = (a\cos t, a\sin t, 0), \qquad 0 \leq t \leq 2\pi.$$

Thus

$$\mathbf{r}'(t) = (-a\sin t, a\cos t, 0),$$

while along C, $\mathbf{f} = (x^2y, z, 0) = (a^3\cos^2 t\sin t, 0, 0)$ and the line integral is therefore

$$\begin{aligned}\oint_C \mathbf{f}\cdot\mathbf{dr} = \oint_C \mathbf{f}\cdot\mathbf{r}'(t)dt &= \int_0^{2\pi}(a^3\cos^2 t\sin t, 0, 0)\cdot(-a\sin t, a\cos t, 0)\,dt \\ &= -a^4\int_0^{2\pi}\cos^2 t\sin^2 t\,dt = -a^4\left[\frac{t}{8} - \frac{\sin 4t}{32}\right]_0^{2\pi} = -\frac{\pi a^4}{4}.\end{aligned}$$

(ii) *Evaluate the surface integral*: The hemispherical surface in Figure 3.20 is given parametrically by

$$\mathbf{r}(\theta,\phi) = a(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad 0 \leq \theta \leq \pi/2, \quad 0 \leq \phi \leq 2\pi.$$

(Recall that this can be obtained from the definition of spherical co-ordinates bearing in mind that a spherical surface is defined by $r = a$). Thus

$$\mathbf{r}_\theta = a(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta), \qquad \mathbf{r}_\phi = a(-\sin\theta\sin\phi, \sin\theta\sin\phi, 0),$$

and so

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = a^2(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \cos\theta\sin\theta).$$

To evaluate the surface integral we use (3.22). Now

$$\nabla\times\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & z & 0 \end{vmatrix} = (-1, 0, -x^2) = (-1, 0, -a^2\sin^2\theta\cos^2\phi).$$

Hence (3.22) becomes

$$
\begin{aligned}
\iint_S (\nabla \times \mathbf{f}) \cdot \mathbf{dS} &= \iint_R (\nabla \times \mathbf{f})(\theta, \phi) \cdot (\mathbf{r}_\theta \times \mathbf{r}_\phi)\, d\theta\, d\phi \\
&= -a^2 \int_0^{2\pi} \int_0^{\pi/2} \left[\sin^2\theta \cos\phi + a^2 \sin^3\theta \cos\theta \cos^2\phi\right] d\theta\, d\phi \\
&= -a^2 \int_0^{2\pi} \cos\phi \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} d\phi - a^4 \int_0^{2\pi} \cos^2\phi \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2} d\phi \\
&= -\frac{a^2\pi}{4} \int_0^{2\pi} \cos\phi\, d\phi - \frac{a^4}{4} \int_0^{2\pi} \cos^2\phi\, d\phi \\
&= 0 - \frac{a^4}{4}\left[\frac{\phi}{2} - \frac{1}{4}\sin 2\phi\right]_0^{2\pi} = -\frac{\pi a^4}{4}.
\end{aligned}
$$

Note:

1. In the above example we used the following indefinite integrals:

$$
\int \cos^2 t \sin^2 t\, dt = \frac{t}{8} - \frac{1}{32}\sin 4t, \qquad \int \sin^2 t\, dt = \frac{t}{2} - \frac{1}{4}\sin 2t
$$

$$
\int \cos t \sin^3 t\, dt = \frac{1}{4}\sin^4 t, \qquad \int \cos^2 t\, dt = \frac{t}{2} + \frac{1}{4}\sin 2t.
$$

2. It is possible to take a shortcut in carrying out part (ii). On referring to Figure 3.20, note that the theorem could also be applied to the surface in the xy-plane consisting of the solid disc $x^2 + y^2 \leq a^2$. In Stokes' theorem, it does not matter which open surface we use as long as it is open and bounded by the curve C. Thus if we use the disc in the xy-plane (label it D) then for this disc $\mathbf{dS} = \mathbf{k}\, dx\, dy$ as it is a planar area and the **outer** normal is clearly in the direction $\mathbf{k}$ (i.e. the positive z-axis). Thus we obtain

$$
\iint_S (\nabla \times \mathbf{f}) \cdot \mathbf{dS} = \iint_D (-1, 0, -x^2) \cdot (0, 0, 1)\, dx\, dy = -\iint_D x^2\, dx\, dy.
$$

This is just an area integral in the xy-plane and is easily evaluated by transforming to polar co-ordinates ($x = r\cos\theta$, $y = r\sin\theta$). Thus (recalling that $\frac{\partial(x,y)}{\partial(r,\theta)} = r$)

$$
-\iint_D x^2\, dx\, dy = -\int_0^{2\pi} \int_0^a r^2 \cos^2\theta\, r\, dr\, d\theta = -\frac{a^4}{4} \int_0^{2\pi} \cos^2\theta\, d\theta = -\frac{\pi a^4}{4}.
$$

Example 3.8.6 *Use Stokes' theorem to evaluate* $\oint_C \mathbf{F} \cdot \mathbf{dr}$ *where* $\mathbf{F} = (z^2, y^2, x)$ *and* C *is the triangle with vertices* $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, *see Figure 3.21.*

Solution. To evaluate the line integral we would have to split C into three line segments, as can be seen from Figure 3.21. Using Stokes' Theorem (3.34) turns out to be much easier. Now

$$
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = (0, 2z - 1, 0).
$$

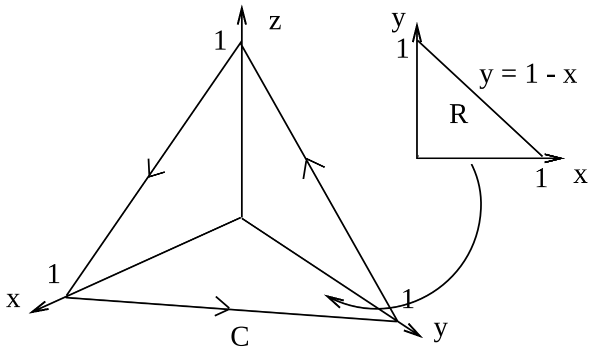

Figure 3.21: *Triangle with vertices* $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

The surface S is the plane $x+y+z=1$ and so $z=g(x,y)=1-x-y$. Hence $\mathbf{r}(x,y)=(x,y,1-x-y)$ and so we use (3.22) with $\mathbf{f}=\nabla\times\mathbf{F}$. We can easily determine

$$\mathbf{r}_x\times\mathbf{r}_y=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -1\\ 0 & 1 & -1\end{vmatrix}=(1,1,1),$$

and

$$\mathbf{f}(\mathbf{r}(x,y))=(\nabla\times\mathbf{F})(\mathbf{r}(x,y))=(0,2(1-x-y)-1,0)=(0,1-2x-2y,0).$$

The region R is given in Figure 3.21 and so, as discussed in § 3.4, if we integrate first w.r.t. y and second w.r.t. x then the region R is described by $0\le y\le 1-x$, $0\le x\le 1$. Hence using (3.34) (and (3.22)) we have

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1\int_0^{1-x}(0,1-2x-2y,0)\cdot(1,1,1)\,dy\,dx\\ &= \int_0^1\int_0^{1-x}(1-2x-2y)\,dy\,dx\\ &= \int_0^1\left[y-2xy-y^2\right]_{y=0}^{y=1-x}dx\\ &= \int_0^1 (x^2-x)\,dx=-\frac{1}{6}.\end{aligned}$$