UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics and Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4006

SEMESTER: Spring 2012

MODULE TITLE: Engineering Mathematics 5

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. Sarah Mitchell

PERCENTAGE OF TOTAL MARKS: 80%

EXTERNAL EXAMINER: Prof. Tim Myers

INSTRUCTIONS TO CANDIDATES:
Answer any **two** questions from Section A and any **two** questions from Section B.
There are some **useful formulae** on the back page of this exam.
You may use a calculator and log tables.

SECTION A
Answer any two questions

1. (a) Show that the first derivative of a unit vector $\hat{\mathbf{u}} = \hat{\mathbf{u}}(t)$ is always perpendicular to $\hat{\mathbf{u}}$ provided the derivative is not zero. 3

   (b) A particle moves along a curve whose parametric equations are

$$x = 2\cos 2t\,, \qquad y = 2\sin 2t\,, \qquad z = 3t\,,$$

   at time t.

   (i) Sketch the curve. 2

   (ii) Determine the particle's velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. Hence show that $\mathbf{v}$ is perpendicular to $\mathbf{a}$. 4

   (iii) Find an expression for the arclength from $t = 0$ to an arbitrary point on the curve. 4

   (iv) Write down the equation for the curve in intrinsic form and find expressions for the curvature and radius of curvature. 5

   (c) Find the directional derivative of the scalar valued function

$$f(x, y, z) = \sin(xz) + \ln y\,,$$

   at the point $(1, 1, \pi)$ in the direction of the vector $\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$. 7

2. (a) Find the work done in moving a particle in a force field given by

$$\mathbf{F}(x, y) = 3y\,\mathbf{i} - 2x^2\,\mathbf{j}\,,$$

   along the curve C in the xy-plane, $y = 3x^3$, from the point $(0, 0)$ to the point $(1, 3)$. 7

   (b) Evaluate the integral $\iint_R xy\,\mathrm{d}x\,\mathrm{d}y$ over the area bounded by the lines $y = x^2$, $x + y = 2$ and the y-axis. 7

   (c) State Stokes' Theorem in the plane.
   Let C be the circle $x^2 + y^2 = 1$ in the plane $z = 2$, oriented counter-clockwise when viewed from above. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where 11

$$\mathbf{F}(x, y, z) = (x - 2z)\,\mathbf{i} + (2x - z^2)\,\mathbf{j} + (x + 3y)\,\mathbf{k}\,.$$

3. (a) Using the fact that $\nabla\Omega$ (where $\Omega = \Omega(x, y)$ is a scalar valued function of position) is perpendicular to its own level curves, find a unit normal to the curve defined by $y - \sqrt{x} + 1 = 0$ at the point $(1, 0)$. Verify this result with a rough sketch. 6

   (b) Show that the vector field

$$\mathbf{v} = (2xy - z^3)\,\mathbf{i} + x^2\,\mathbf{j} - (3xz^2 + 1)\,\mathbf{k}\,,$$

   is conservative. Determine an associated *scalar potential* ϕ for this vector field $\mathbf{v}$. Hence find $\int_C \mathbf{v}\cdot\mathbf{dr}$ where C is any curve starting at the point $(0, 1, 0)$ and ending at the point $(1, 2, 0)$. 11

   (c) Use Taylor's theorem in three dimensions to find a first order approximation for

$$g(x, y, z) = x\sin y - x^2 + z\,,$$

   at the point $(1, \pi/2, 1)$. Hence estimate $g(1.1, 1.5, 0.9)$. 8

SECTION B

Answer any two questions

4. Consider the second order partial differential equation

$$A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} = f(x, y, u, u_x, u_y)\,,$$

where A, B, C and f are all known functions. This equation has characteristics defined by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B \pm \sqrt{B^2 - AC}}{A}\,.$$

   (a) Give the criteria to classify the above equation as elliptic, hyperbolic or parabolic. 3

   (b) Consider the following partial differential equations:

$$\begin{array}{ll} \text{(i)} & u_{xx} + u_x - 3u_{yy} = 0 \\ \text{(ii)} & u_{xx} - 4u_{xy} + 4u_{yy} = e^x \\ \text{(iii)} & u_{xx} + 3u_{tt} = \cos x\,. \end{array}$$

   Fully classify these equations and find and sketch the real characteristics (if any). 9

(c) Find the equation relating the constants k, α, β, and c so that the function

$$u = e^{-ky}\cos(\alpha x)\cos(\beta y)\,,$$

is a solution of the partial differential equation 7

$$c^2 u_{xx} = u_{yy} + 2ku_y\,.$$

(d) Solve the following pseudo differential equations for $u = u(x, y)$: 6

$$xu_x - 1 = 0, \qquad u_y + u^3 = 0\,.$$

5. (a) Consider a bar of heat conducting material of length l. The partial differential equation describing the conduction of heat through the bar is given by

$$u_t = c^2 u_{xx}\,.$$

Assume that the ends of the bar are kept at zero temperature and so

$$u(0, t) = u(l, t) = 0\,.$$

Use the method of separation of variables to show that a form of the solution appropriate to these boundary conditions is given by

$$u(x, t) = \sum_{n=1}^{\infty} D_n \exp\left(-\frac{c^2 n^2 \pi^2 t}{l^2}\right)\sin\left(\frac{n\pi x}{l}\right)\,,$$

where the D_n are constants. 12

Show that if, in addition, $u = \sin(2\pi x/l)$ when $t = 0$ for $0 < x < l$, then the solution simplifies to 4

$$u(x, t) = \exp\left(-\frac{4c^2\pi^2 t}{l^2}\right)\sin\left(\frac{2\pi x}{l}\right)\,.$$

(b) Use Laplace transforms to solve the wave equation $u_{tt} = c^2 u_{xx}$, on the semi-infinite domain $0 \le x < \infty$, $0 \le t < \infty$, subject to $u(x, 0) = 0$, $u_t(x, 0) = 0$, $u(0, t) = t^2$ and $u \to 0$ as $x \to \infty$. 9

6. (a) Consider the finite difference approximation

$$f'(x) \cong \frac{-f(x + 2h) + 4f(x + h) - 3f(x)}{2h}\,.$$

Show that this has $O(h^2)$ accuracy. 5

(b) Let Ω be the unit square $(0,1) \times (0,1)$, with boundary Γ and closure $\bar{\Omega} = [0,1] \times [0,1]$. Consider the boundary value problem

$$Lu = u_{xx} + u_{yy} = f(x,y)\,, \quad (x,y) \in \Omega\,,$$

with boundary conditions

$$u(x,0) = \phi_0(x)\,, \qquad u(x,1) = \phi_1(x)$$
$$u(0,y) = \psi_0(y)\,, \qquad u(1,y) = \psi_1(y)\,.$$

Taking a fixed-width mesh h in both the x and y directions, formulate the discretised problem and outline a uniform mesh. The second derivative can be approximated using the standard $O(h^2)$ operator. 5

(c) Using the result from part (b), determine the linear system of equations resulting from the discretised problem, with $f(x,y) = e^{x+y}$, and boundary conditions given by

$$u(x,0) = x(1-x)\,, \qquad u(x,1) = 1$$
$$u(0,y) = y\,, \qquad u(1,y) = y\,.$$

Use a stepsize of $h = \frac{1}{3}$ in both the x and y directions. 8

(d) Consider the heat equation $u_t = c^2 u_{xx}$ for a bar of conducting material of length 1, subject to

$$u(x,0) = x^2 - 2x + 1\,, \qquad u(0,t) = 0\,, \qquad \frac{\partial u}{\partial x}(1,t) = 0\,.$$

Formulate the discretised problem using an explicit finite difference method on a uniform mesh (taking central differences to approximate the second derivative). 7

Useful Information

- Taylor Series in more than one variable:

$$f(x, y, z) = f(x_0, y_0, z_0) + \delta\mathbf{r} \cdot \nabla f|_{(x_0,y_0,z_0)} + O(|\delta\mathbf{r}|^2)\,.$$

  where $\delta\mathbf{r} = (h, k, l)$ with $h = x - x_0$, $k = y - y_0$ and $l = z - z_0$.

- Half-range Fourier **sine** series of period $2l$:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$\text{with} \quad B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x\,.$$

- Useful Laplace transform results. Suppose $\mathbf{L}(f) = \hat{f}(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$. Then the t *shift* property states that

$$\mathbf{L}^{-1}\left[e^{-as}\hat{f}(s)\right] = H(t-a)f(t-a)\,.$$

  Also

$$\hat{f}(s) = \frac{1}{s^3} \qquad \Longrightarrow \qquad f(t) = \frac{t^2}{2}\,.$$