MA4006: Spring 2011 Exam solutions

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SECTION A

Question 1

(a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2+2 & 2xyz-1 \end{vmatrix} = \mathbf{i}\,(2xz-2xz) - \mathbf{j}\,(2yz-2yz) + \mathbf{k}\,(z^2-z^2) = \mathbf{0}.$$

Hence $\mathbf{F}$ is conservative and so there exists ϕ such that $\mathbf{F} = \nabla\phi$, i.e.

$$(yz^2, xz^2+2, 2xyz-1) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right),$$

or

$$\text{(i)}\quad \frac{\partial\phi}{\partial x} = yz^2, \qquad \text{(ii)}\quad \frac{\partial\phi}{\partial y} = xz^2+2, \qquad \text{(iii)}\quad \frac{\partial\phi}{\partial z} = 2xyz-1.$$

(i) $\frac{\partial\phi}{\partial x} = yz^2$ and so

$$\phi = xyz^2 + f(y,z) \qquad \Longrightarrow \qquad \frac{\partial\phi}{\partial y} = xz^2 + \frac{\partial f}{\partial y}. \tag{1}$$

(ii) Also $\frac{\partial\phi}{\partial y} = xz^2+2$ and so comparing with (1) we see that $\frac{\partial f}{\partial y} = 2$, or $f = 2y + g(z)$ and so $\phi = xyz^2 + 2y + g(z)$.

(iii) Finally, $\frac{\partial\phi}{\partial z} = 2xyz + g'(z)$ and so we deduce that $g'(z) = -1$, or $g(z) = -z + c$, where c is an arbitrary constant..

Thus the scalar potential is $\phi(x,y,z) = xyz^2 + 2y - z + c$. Using the fundamental theorem of line integrals we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla\phi\,\cdot d\mathbf{r} = \phi(1,2,1) - \phi(1,1,0) = (2+4-1+c) - (2+c) = 3.$$

(b) The level surfaces of $\Omega(x,y,z)$ are those surfaces defined by

$$\Omega(x,y,z) = c, \qquad \text{where } c \text{ is constant.}$$

$\nabla\Omega$ is normal to the level surface $\Omega(x,y,z) = c$. Consider $z = x^2\cos y$. We can write this as $\Omega(x,y,z) = 0$ where

$$\Omega(x,y,z) = z - x^2\cos y.$$

Then

$$\nabla\Omega = -2x\cos y\,\mathbf{i} + x^2 \sin y\,\mathbf{j} + \mathbf{k}.$$

Thus at the point $(1, \pi/2, 0)$:

$$\nabla\Omega = \mathbf{j} + \mathbf{k},$$

and so the unit normal is

$$\frac{1}{\sqrt{1^2+1^2}}(\mathbf{j}+\mathbf{k}) = \frac{1}{\sqrt{2}}(0,1,1).$$

(c) The unit vector in the direction $(2, 0, -1)$ is

$$\frac{(2,0,-1)}{\sqrt{2^2+(-1)^2}} = \frac{1}{\sqrt{5}}(2,0,-1).$$

Now

$$\nabla f = \left(-\frac{2x}{x^2+y^2+z^2}, -\frac{2y}{x^2+y^2+z^2}, -\frac{2z}{x^2+y^2+z^2}\right) \quad \Rightarrow \quad \nabla f\big|_{(-1,0,1)} = (1,0,-1).$$

Thus

$$\frac{\partial f}{\partial n} = (1,0,-1)\cdot\frac{1}{\sqrt{5}}(2,0,-1) = \frac{3}{\sqrt{5}}.$$

Question 2

(a) The right half of the circle can be parameterised as

$$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j}, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

We use

$$\int_C f(x,y)\,ds = \int_{-\pi/2}^{\pi/2} f(\mathbf{r}(t))|\mathbf{r}'(t)|dt,$$

where $f = xy^4 = 1024\cos t\sin^4 t$. Now,

$$\mathbf{r}'(t) = -4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} \qquad \Rightarrow \qquad |\mathbf{r}'(t)| = 4,$$

and so

$$\int_C xy^4\,ds = 4096\int_{-\pi/2}^{\pi/2}\cos t\sin^4 t\,dt = 4096\left[\frac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \frac{8192}{5}.$$

(b) The surface area A is given by

$$A = \iint_S dS = \iint_R |\mathbf{r}_r \times \mathbf{r}_\theta|\,dr\,d\theta.$$

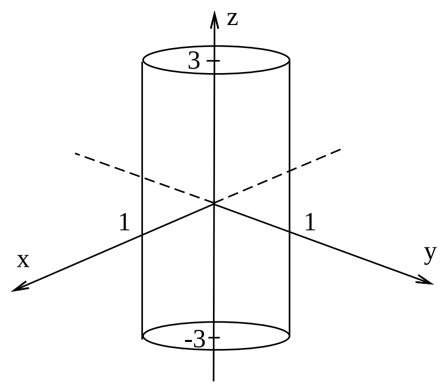
Now

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta, -r\sin\theta, r).$$

Thus $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r$ and so the surface area is

$$A = \int_0^{2\pi} \int_0^1 \sqrt{2}r \, dr \, d\theta = \frac{\sqrt{2}}{2} \int_0^{2\pi} r \, d\theta = \sqrt{2}\pi.$$

(c) (i) Sketch is



(ii) The divergence theorem: Consider a **closed** region V bounded by a piecewise smooth closed surface S. If the vector $\mathbf{f}$ is defined and continuously differentiable throughout V then

$$\iint_S \mathbf{f} \cdot \mathbf{dS} = \iiint_V \nabla \cdot \mathbf{f} \, dV.$$

(iii) Now

$$\nabla \cdot \mathbf{f} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(x^3, y^3, \cos(x^3 y^3) \right) = 3x^2 + 3y^2.$$

Use cylindrical co-ordinates [From useful information result given at end of paper]:

$$\mathbf{r}(r, \theta, z) = (r\cos\theta, r\sin\theta, z), \qquad 0 \le \theta \le 2\pi, \quad -3 \le z \le 3.$$

Since $x^2 + y^2 \le 4$ it follows that $0 \le r \le 2$. Then

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r, \qquad (\nabla \cdot \mathbf{f})(\mathbf{r}(r, \theta, z)) = 3r^2.$$

Thus

$$\iiint_V \nabla \cdot \mathbf{f} \, dV = 3 \int_{-3}^{3} \int_0^{2\pi} \int_0^2 r^3 \, dr \, d\theta \, dz = 12 \int_{-3}^{3} \int_0^{2\pi} d\theta \, dz = 24\pi \int_{-3}^{3} dz = 144\pi.$$

Question 3

(a) Differentiating $\mathbf{r}(t)$ gives

$$\mathbf{r}'(t) = \begin{cases} (2t, -4\sin t, 2e^{2t}), & \text{if } -1 \le t \le 0 \\ (2, -1, e^{-t}), & \text{if } 0 < t \le 2. \end{cases}$$

Thus

$$|\mathbf{r}'(t)| = \begin{cases} \sqrt{4t^2 + 16\sin^2 t + 4e^{4t}}, & \text{if } -1 \le t \le 0 \\ \sqrt{4 + (-1)^2 + e^{-2t}}, & \text{if } 0 < t \le 2. \end{cases}$$

and so

$$\hat{\mathbf{t}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \begin{cases} \frac{(2t,-4\sin t, 2e^t)}{\sqrt{4t^2+16\sin^2 t+4e^{4t}}}, & \text{if } -1 \le t \le 0 \\ \frac{(2,-1,e^{-t})}{\sqrt{5+e^{-2t}}}, & \text{if } 0 < t \le 2. \end{cases}$$

As $t \to 0^+$, $\mathbf{r}(t) = (0, 4, 1)$ and $\mathbf{r}'(t) = (0, 0, 2)$ and as $t \to 0^-$, $\mathbf{r}(t) = (0, 4, 1)$ and $\mathbf{r}'(t) = (2, -1, 1)$. Hence the curve is continuous but not smooth at $t = 0$, but smooth elsewhere, and so piecewise smooth.

(b) $\mathbf{r}'(t) = (-a\sin t, a\cos t, c)$ and so

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2 + c^2}.$$

Hence

$$s = \int_0^t |\mathbf{r}'(t_0)|\, dt_0 = \int_0^t \sqrt{a^2 + c^2}\, dt_0 = kt,$$

where $k = \sqrt{a^2 + c^2}$. Thus $t = s/k$ and so the intrinsic equation can be written as

$$\mathbf{r}(s) = (a\cos(s/k), a\sin(s/k), cs/k).$$

The curvature $\kappa(s)$ is given by

$$\kappa(s) = \left|\mathbf{r}''(s)\right|,$$

and so

$$\begin{aligned} \mathbf{r}'(s) &= \left(-\frac{a}{k}\sin(s/k), \frac{a}{k}\cos(s/k), \frac{1}{k}\right) \\ \mathbf{r}''(s) &= \left(-\frac{a}{k^2}\cos(s/k), -\frac{a}{k^2}\sin(s/k), 0\right) \\ \Rightarrow \quad \kappa(s) &= \sqrt{\left(-\frac{a}{k^2}\cos(s/k)\right)^2 + \left(-\frac{a}{k^2}\sin(s/k)\right)^2} = \frac{a}{k^2}. \end{aligned}$$

(c) Taylor's series is [From useful information result given at end of paper]:

$$f(x, y, z) = f(x_0, y_0, z_0) + \delta\mathbf{r} \cdot \nabla f\big|_{(x_0,y_0,z_0)} + O\left(|\delta\mathbf{r}|^2\right).$$

Here $(x_0, y_0, y_0) = (1, 1, 2)$ and

$$\delta\mathbf{r} = (h, k, l) = (x - x_0, y - y_0, z - z_0) = (x - 1, y - 1, z - 2),$$

and

$$\nabla f = (y^2, 2xy + z, y) \qquad \Longrightarrow \qquad \nabla f\big|_{(1,1,2)} = (1, 4, 1).$$

Also $f(1, 1, 2) = 3$. Hence

$$f(x, y, z) = 3 + (x - 1, y - 1, z - 2) \cdot (1, 4, 1) + O\big(|\delta\mathbf{r}|^2\big) = -4 + x + 4y + z + O\big(|\delta\mathbf{r}|^2\big).$$

If $(x, y) = (0.9, 1.2, 2.3)$ then

$$\delta\mathbf{r} = (-0.1, 0.2, 0.3) \qquad \Rightarrow \qquad |\delta\mathbf{r}|^2 = 0.01 + 0.04 + 0.09 = 0.14.$$

Hence

$$f(0.9, 1.2, 2.3) = -4 + 0.9 + 4(1.2) + 2.3 + O(0.14) = 4 + O(0.14).$$

SECTION B

Question 4

(a) A hyperbolic equation satisfies $B^2 - AC > 0$, a parabolic equation satisfies $B^2 - AC = 0$ and an elliptic equation satisfies $B^2 - AC < 0$.

(b) (i) Here $A = 4$, $B = C = 0$. Thus $B^2 - AC = 0$ and so the PDE is parabolic. There is only one set of characteristics which satisfy

$$\frac{dt}{dx} = \frac{B}{A} = 0$$

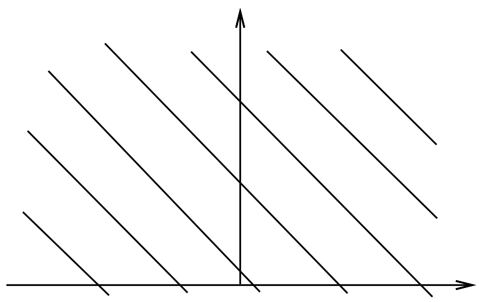
or $t = c$, where c is an arbitrary constant. Characteristics are just horizontal lines in the t against x graph.

(b) (ii) Here $A = 1$, $B = 0.5$ and $C = 3$. Thus $B^2 - AC = (0.5)^2 - 1(3) < 0$ and so the PDE is elliptic. Thus there are no characteristics.

(b) (iii) Here $A = 1$, $B = -3$ and $C = 1$. Thus $B^2 - AC = (-3)^2 - 1(1) = 8 > 0$ and so the PDE is hyperbolic. The characteristics are given by

$$\frac{d\theta}{dr} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{-3 \pm \sqrt{8}}{1} = -3 \pm 2\sqrt{2} \qquad \Longrightarrow \qquad \frac{d\theta}{dr} = -3+2\sqrt{2}, \quad \text{or} \quad \frac{d\theta}{dr} = -3-2\sqrt{2},$$

or $\theta = (-3 + 2\sqrt{2})r + c$ and $\theta = (-3 - 2\sqrt{2})r + c$. The slope of both is negative and so the characteristics look like this:



(c) Consider $\xi = x + 6y$, $\eta = x - y$. Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

and so $u_x = u_\xi + u_\eta$. This means

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

and so $u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$. Also

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = 6\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta},$$

and so $u_y = 6u_\xi - u_\eta$. This means

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \xi}\left(6\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(6\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y} = 36\frac{\partial^2 u}{\partial \xi^2} - 12\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

and so $u_{yy} = 36u_{\xi\xi} - 12u_{\xi\eta} + u_{\eta\eta}$. Also

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial \xi}\left(6\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(6\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = 6\frac{\partial^2 u}{\partial \xi^2} + 5\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial^2 u}{\partial \eta^2},$$

and so $u_{xy} = 6u_{\xi\xi} + 5u_{\xi\eta} - u_{\eta\eta}$. Thus $6u_{xx} + 5u_{xy} - u_{yy} = 0$ becomes

$$6(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) + 5(6u_{\xi\xi} + 5u_{\xi\eta} - u_{\eta\eta}) - (36u_{\xi\xi} - 12u_{\xi\eta} + u_{\eta\eta}) = 0$$

or

$$u_{\xi\eta} = 0 \qquad \Longrightarrow \qquad u = f(\xi) + g(\eta).$$

So

$$u(x,y) = f(x+6y) + g(x-y) \tag{2}$$

$$u_y(x,y) = 6f'(x+6y) - g'(x-y). \tag{3}$$

From the conditions $u(x,0) = \sin(2x)$ and $u_y(x,0) = \cos(2x)$ we use (2) and (3) to deduce that

$$f(x) + g(x) = \sin(2x), \qquad 6f'(x) - g'(x) = \cos(2x).$$

Thus $6f(x) - g(x) = \frac{1}{2}\sin(2x) + k$ where k is an arbitrary constant. Solving these two equations gives

$$f(x) = \frac{3}{14}\sin(2x) + \frac{1}{7}k, \qquad g(x) = \frac{11}{14}\sin(2x) - \frac{1}{7}k.$$

Finally we have

$$u(x,y) = \frac{3}{14}\sin[2(x+6y)] + \frac{11}{14}\sin[2(x-y)].$$

Question 5

(a) We must solve $u_{xx} + u_{yy} = 0$ on $0 < x < l$, $y > 0$, using the method of separation of variables subject to the boundary conditions $u(0,y) = 0$, $u(l,y) = 0$, $u(x,0) = u_0$ and

$u(x,y) \to 0$ as $y \to \infty$. Let $u(x,y) = F(x)G(y)$. Then $u_{xx} = F''(x)G(y)$ and $u_{yy} = F(x)\ddot{G}(y)$. Substituting these into $u_{xx} + u_{yy} = 0$ gives

$$F''(x)G(y) + F(x)\ddot{G}(y) = 0 \qquad \Longrightarrow \qquad -\frac{\ddot{G}}{G} = \frac{F''}{F} = k,$$

or

$$\ddot{G} + kG = 0, \qquad F'' - kF = 0.$$

The boundary conditions $u(0,y) = u(l,y) = 0$ therefore imply $F(0) = F(l) = 0$.

- $k = 0$: The solution of $F'' = 0$ is $F(x) = Ax + B$. The bc $F(0) = F(l) = 0$ imply that $A = B = 0$ and so there are no non-trivial solutions.

- $k = \mu^2 > 0$: The solution of $F'' - \mu^2 F = 0$ is $F(x) = Ae^{\mu x} + Be^{-\mu x}$. The bc $F(0) = 0$ implies $A + B = 0$ and the bc $F(l) = 0$ implies $Ae^{\mu l} + Be^{-\mu l} = 0$. Then $A = B = 0$ and so there are no non-trivial solutions.

- $k = -p^2 < 0$: The solution of $F'' + p^2 F = 0$ is $F(x) = A\cos px + B\sin px$. The bc $F(0) = 0$ implies $A = 0$ and the bc $F(l) = 0$ implies $B\sin(pl) = 0$, and so $pl = n\pi$. Thus
$$F_n(x) = B_n \sin\frac{n\pi x}{l}.$$

Now solve the ODE for G. We have

$$\ddot{G}_n - \frac{n^2\pi^2}{l^2}G_n = 0 \qquad \Longrightarrow \qquad G_n(y) = C_n e^{\lambda_n y} + D_n e^{-\lambda_n y},$$

where $\lambda_n = n\pi/l$. The bc $u(x,y) \to 0$ as $y \to \infty$ implies that $G(y) \to 0$ as $y \to \infty$ and so $C_n = 0$. Hence

$$u_n(x,y) = F_n(x)G_n(y) = E_n e^{-\lambda_n y}\sin\frac{n\pi x}{l},$$

where $E_n = B_n D_n$. This gives

$$u(x,y) = \sum_{n=1}^{\infty} E_n e^{-\lambda_n y}\sin\frac{n\pi x}{l}.$$

Finally, to find E_n we use the bc $u(x,0) = u_0$. Thus

$$u_0 = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \qquad \Rightarrow \qquad E_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}dx = \begin{cases} \frac{4u_0}{n\pi} & n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

[From useful information result given at end of paper].

(b) Let $\hat{u}(x,s) = \int_0^\infty u(x,t)e^{-st}dt$. Then

$$\begin{aligned}\int_0^\infty u_t(x,t)e^{-st}dt &= \left[u(x,t)e^{-st}\right]_0^\infty + s\int_0^\infty u(x,t)e^{-st}dt \\ &= -u(x,0) + s\int_0^\infty u(x,t)e^{-st}dt \\ &= s\hat{u}(x,s),\end{aligned}$$

since $u(x,0) = 0$. Also

$$\int_0^\infty u_x(x,t)e^{-st}dt = \frac{d}{dx}\int_0^\infty u(x,t)e^{-st}dt = \hat{u}_{xx}.$$

Hence the PDE becomes

$$\hat{u}_{xx} - s\hat{u} = 0 \qquad \Longrightarrow \qquad \hat{u}(x,s) = Ae^{-\sqrt{s}x} + Be^{\sqrt{s}x}.$$

The BC $u(0,t) = t$ implies that

$$\hat{u}(0,s) = \int_0^\infty e^{-st}u(0,t)dt = \int_0^\infty te^{-st}dt = \frac{1}{s^2}.$$

From the bc $u \to 0$ as $x \to \infty$ it follows that $\hat{u} \to 0$ as $x \to \infty$ and so $B = 0$. Also $\hat{u}(0,s) = 1/s^2$ implies that $A = 1/s^2$. Thus

$$\hat{u}(x,s) = \frac{1}{s^2}e^{-\sqrt{s}x} \qquad \Longrightarrow \qquad u(x,t) = \left(t + \frac{x^2}{2}\right)\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \sqrt{\frac{t}{\pi}}xe^{-x^2/[4t]}.$$

[From useful information result given at end of paper].

(c) If $u = e^{\alpha t}(\theta - \theta_0)$ then

$$u_t = \alpha e^{\alpha t}(\theta - \theta_0) + e^{\alpha t}\theta_t, \qquad u_x = e^{\alpha t}\theta_x, \qquad u_{xx} = e^{\alpha t}\theta_{xx}.$$

So

$$u_t - c^2u_{xx} = \alpha e^{\alpha t}(\theta - \theta_0) + e^{\alpha t}\theta_t - c^2e^{\alpha t}\theta_{xx} = e^{\alpha t}[\theta_t - c^2\theta_{xx} + \alpha(\theta - \theta_0)] = 0,$$

since $\theta_t = c^2\theta_{xx} - \alpha(\theta - \theta_0)$.

Question 6

(a) Use Taylor Series:

$$\begin{aligned}f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f''''(x) + \ldots \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f''''(x) + \ldots.\end{aligned}$$

Thus

$$
\begin{aligned}
f''(x) &\cong \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \\
&= \frac{1}{h^2}\left\{\left[f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f''''(x) + \dots\right]\right. \\
&\quad -2f(x) \\
&\quad \left.+\left[f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f''''(x) + \dots\right]\right\} \\
&= \frac{1}{h^2}\left\{h^2 f''(x) + \frac{1}{12}h^4 f'''(x) + \dots\right\} \\
&= f''(x) + O(h^2),
\end{aligned}
$$

and so the finite difference approximation has $O(h^2)$ accuracy.

(b) Use central differences for the second derivatives, with $\Delta x = \Delta y = h$. Then

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}, \qquad \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}.$$

Hence the PDE $u_{xx} + u_{yy} = f(x,y)$ is discretised as

$$u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y) \approx h^2 f(x,y). \tag{4}$$

The square is discretised by a uniform mesh of width $h = 1/N$. The mesh points are the intersections of the lines (x_i, y_j) where $x_i = ih$ and $y_j = jh$ for some $0 \le i,\, j \le N$. Writing the approximation to $u(x_i, y_j)$ as $u_{i,j}$, and similarly for f, equation (4) becomes

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}, \qquad 1 \le i,\, j \le N-1. \tag{5}$$

On the boundary we have

$$u_{i,0} = \phi_0(x_i), \qquad u_{i,N} = \phi_1(x_i), \qquad u_{0,j} = \psi_0(y_j), \qquad u_{N,j} = \psi_1(y_j).$$

(c) Using $h = 1/3$ means there are just 4 internal points. Since $f = -1$ the difference equation (5) becomes

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -h^2, \qquad 1 \le i,\, j \le 2.$$

The boundary conditions are

$$u_{i,0} = 0, \qquad u_{i,N} = x_i, \qquad i = 0,\, 1,\, 2,\, 3$$

$$u_{0,j} = 0, \qquad u_{N,j} = y_j^2, \qquad j = 0,\, 1,\, 2,\, 3.$$

Hence the linear system of equations to solve is

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = -h^2$$
$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} - 4u_{2,1} = -h^2$$
$$u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 4u_{1,2} = -h^2$$
$$u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} - 4u_{2,2} = -h^2.$$

This determines the unknown interior points $u_{1,1}$, $u_{2,1}$, $u_{1,2}$ and $u_{2,2}$.

(d) Let $u_{i,j} = u(x_i, t_j)$, where $x_i = i\Delta x$, $i = 0, 1, \ldots, I$ and $t_j = j\Delta t$, $j = 0, 1, 2, \ldots$. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \qquad \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2}.$$

Putting these into $u_{tt} = c^2 u_{xx}$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{c^2 \Delta t^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

or, on rearranging,

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \lambda^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}), \tag{6}$$

where $\lambda = c\Delta t/\Delta x$, $\Delta x = 1/I$ and $i = 1, \ldots, I-1$, $j = 1, 2, \ldots$. The boundary conditions $u(0,t) = u(1,t) = 0$ imply $u_{0,j} = u_{I,j} = 0$. The initial condition $u(x,0) = x(1-x)$ implies $u_{i,0} = x_i(1-x_i)$, $i = 0, 1, \ldots, I$. For the initial condition $u_t(x,0) = 0$ we use a central difference to deduce that

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} \equiv \frac{u_{i,1} - u_{i,-1}}{2\Delta t} = 0 \qquad \Longrightarrow \qquad u_{i,1} = u_{i,-1}.$$

Then from (6) with $j = 0$ we find that

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + \lambda^2(u_{i+1,0} - 2u_{i,0} + u_{i-1,0}),$$

or

$$u_{i,1} = (1-\lambda^2)u_{i,0} + \frac{\lambda^2}{2}(u_{i+1,0} - u_{i-1,0}).$$