UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4006

SEMESTER: Spring 2006

MODULE TITLE: Engineering Mathematics 5

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. Helen Purtill

PERCENTAGE OF TOTAL MARKS: 100%

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Answer any 4 questions.**

1. (a) Consider the vector valued function

$$\mathbf{r}(t) = \begin{cases} (t^2,\ 4\cos(\pi t),\ -e^{3t}) & \text{if } -1 \le t \le 0 \\ (2t,\ 4-t,\ 3e^{-t}-4) & \text{if } \ 0 \le t \le 2 \end{cases}$$

find its derivative, and hence find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ and comment on its smoothness 10

(b) Find an expression for the arclength of the curve expressed parametrically as $\mathbf{r}(t) = (\cos 2t, -\sin 2t, t)$ where $0 \le t \le 2\pi$. Using the arclength find the intrinsic equation of the curve represented by $\mathbf{r}(\mathbf{t})$. Find an expression for the curvature of this curve at any point. 7

(c) A particle P moving in a circle of radius R, with constant velocity has position vector $\mathbf{r}(t) = R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j}$ where ω is a known constant and t is the time. Find the acceleration and velocity of the particle and show that the acceleration is towards the centre of the circle. 8

2. (a) Find the work done in moving a particle in a force field given by

$$\mathbf{F}(x, y, z) = 2x^3y\mathbf{i} + yz\mathbf{j} - 3xz^2\mathbf{k},$$

along the curve with parametric definition $\mathbf{r} = (-1, t, t^2)$ where $0 \le t \le 1$. 10

(b) Evaluate the integral $\int\int y\,dx\,dy$ over the area bounded by $y = x^2$, $x + y = 2$ and the y-axis. 6

(c) State the Divergence Theorem in the plane and use it to express the surface integral $\int\int_S \mathbf{f}\cdot d\mathbf{S}$ as a volume integral where $\mathbf{f} = (x^3, y^3, z)$, and S is the surface of the cylinder bounded by $x^2 + y^2 = 1$, $z = 0$, and $z = 3$. Setup the volume integral with proper limits but do not evaluate it. 9

3. (a) Find the directional derivative of the scalar valued function

$$f(x, y, z) = xy^2 + yz$$

at the point $(1, 1, 2)$ in the direction of $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. 7

(b) Find the direction of the line normal to the surface

$$x^2y + y^2z + z^2x + 1 = 0$$

at the point $(1, 2 - 1)$. 5

(c) Show that the vector field $\mathbf{v} = (2xy - z^3)\mathbf{i} + x^2\mathbf{j} - (3xz^2 + 1)\mathbf{k}$ is a conservative vector field. Determine an associated *scalar potential* ϕ for this vector field $\mathbf{v}$. 6

(d) Use Taylor's theorem in two dimensions to find a first order approximation for $f(1.5, 2.2)$ based on quantities evaluated at the point $(1, 2.5)$ in the case where

$$f(x, y) = x \sin y - x^2.$$

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4. (a) If u_1 and u_2 are any solutions of a linear homogeneous partial differential equation $L[u] = 0$ in some region, prove that $u = c_1u_1 + c_2u_2$ is also a solution, where c_1 and c_2 are arbitrary constants. 4

(b) Consider the partial differential equation

$$A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} = f(x, y, u_x, u_y)$$

where A, B, C and f are all known functions. Explain the idea of *characteristics* of a partial differential equation. Noting that the equation defined above has characteristics defined by

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

suggest a way to classify equations as parabolic, elliptic or hyperbolic. Fully classify the following equations. Find and sketch the real characteristics (if any) of the following partial differential equations:

(i) $u_t = 2u_{xx} - u_x$

(ii) $u_{rr} - 4u_{r\theta} + u_{\theta\theta} = (\sin^2\theta)u$

(iii) $yu_{xx} + 2u_{xy} = 0$

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(c) Solve the following pseudo differential equations for $u = u(x, y)$:

(i) $u_x - xu = 0$

(ii) $u_{xx} - 4 = 0$

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(d) Verify that $u(x, y) = e^{-4t} \cos 3x$ is a solution of the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

for a suitable value of the constant c. Identify this value of c. 4

5. (a) A thin bar of copper of length l is heated in an oven till its temperature u is of the form $u = f(x)$, $0 \le x \le l$. The ends of the bar are then placed in ice water maintained at 0^oC.

Assume that the flow of heat in the region $0 \le x \le l$ can be modelled by linear heat equation $u_t = c^2 u_{xx}$. Set up the appropriate initial boundary value problem. Use the method of separation of variables to solve this problem. 13

(b) Consider the equation

$$2u_{xx} - 3u_{xy} + u_{yy} = 0$$

Using the transformation $\alpha = x + 2y$, $\beta = x + y$ reduce the equation to a simpler form and determine the particular solution that satisfies the boundary conditions

$$u(x, 0) = \cos x, \quad u_y(x, 0) = 0.$$

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6. Let Ω be the unit square $(0, 1) \times (0, 1)$, with boundary Γ and closure $\bar{\Omega} = [0, 1] \times [0, 1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega$$

with boundary conditions

$$u(x, y) = g(x, y), \quad (x, y) \in \Gamma.$$

(a) Taking a fixed-width mesh h in both the x and y directions, formulate the discretised problem, $L^h u_{i,j} = f_{i,j}$, and outline a uniform mesh for this problem. The 2nd derivative can be approximated using the standard $O(h^2)$ operator. 4

(b) Show that the finite difference operator L^h satisfies a maximum principle of the form:

$$L^h v_{i,j} \geq 0 \; \forall \, i,j \;\; \Rightarrow \; \max_{\Omega^h} v_{i,j} \leq \max_{\Gamma^h} v_{i,j}$$

where $v_{i,j}$ is any mesh-function defined on $\bar{\Omega}^h$. 8

(c) Explain how an estimate of the upper bound for any mesh function $v_{i,j}$ of the form

$$\max_{\Omega^h} |v_{i,j}| \leq \max_{\Gamma^h} |v_{i,j}| + \frac{1}{2} \max_{\Omega^h} |L^h v_{i,j}|$$

can be employed to comment on the stability and convergence of the numerical method. 5

(d) Taking a fixed-width mesh $h = 1/3$, in both the x and y directions, determine the linear system of equations resulting from the discretised problem where $f(x,y) = x + y$ and the boundary conditions are given by:

$$\begin{array}{ll} u(x,0) = x(1-x), & u(x,1) = 1, \\ u(0,y) = y, & u(1,y) = y. \end{array}$$

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